# REVISITING THE EFFECTS OF MEAN VELOCITY PROFILES SHAPE ON SOUND PROPAGATION THROUGH TWO-DIMENSIONAL DUCTS

**Lucas A. Bonomo**[1*] **Julio A. Cordioli**[1] **Edward J. Brambley**[2]
[1] Federal University of Santa Catarina, Florianópolis, Brazil
[2] University of Warwick, Coventry, United Kingdom

## ABSTRACT

Previous publications suggest that the acoustic attenuation rate in lined two-dimensional ducts with grazing flow depends on the shape of the boundary layer's mean velocity profile. Nayfeh et al. (1974) showed that for downstream-propagating waves, similar attenuation rates can be achieved for different profile shapes if the boundary layer displacement thickness remains constant. However, for upstream propagation, the attenuation rate also varies with the boundary layer shape factor. This study revisits Nayfeh et al.'s work, focusing on small ducts, particularly within the typical Helmholtz number range used in acoustic liner impedance eduction test rigs. Three profile shapes are considered: the sinusoidal flow profile, the hyperbolic tangent profile, and the universal law of the wall, which represents a more realistic turbulent boundary layer. The exact wavenumbers from the Pridmore–Brown equation for these flow profiles are compared with the approximation obtained using a uniform flow and the Ingard–Myers boundary condition applied to the lined wall with slipping grazing velocity.

**Keywords:** *duct acoustics, acoustic liners, sheared flow profile, impedance eduction.*

## 1. INTRODUCTION

Acoustic liners are essential noise-reducing treatments applied to the nacelles of turbofan engines. They typically consist of a perforated facesheet, a honeycomb core, and a hard backplate. These liners are characterised by their acoustic impedance, $\tilde{Z}(\omega) = \theta + \mathrm{i}\chi$, where $\theta$ is the resistance and $\chi$ is the reactance. The impedance, which depends on both liner geometry and operational conditions such as grazing flow velocity and incident Sound Pressure Level (SPL), serves as a boundary condition in noise propagation models, enabling accurate noise predictions while circumventing the computational cost of explicitly modelling the liner.

Liner impedance is typically characterised using experimental methods under conditions as close as possible as those in turbofan engines, with impedance eduction being the most widely used approach [1]. These techniques often assume uniform mean flow and apply the Ingard–Myers boundary condition to describe liner interaction with acoustic and flow fields. However, recent findings challenge these assumptions.

One key issue is the dependence of educed impedance on the wave propagation direction relative to the mean flow, which contradicts the locally reacting hypothesis. This discrepancy has been consistently observed across different experimental facilities, suggesting a fundamental limitation in the standard modelling approach. Initial studies [2] attributed this behaviour to a failure of the Ingard–Myers boundary condition, prompting alternative formulations that introduce additional degrees of freedom, such as viscosity [3], shear stress [4, 5], or boundary layer effects [6]. These modifications aim to remove the directional dependence of the educed impedance. However, experimental validation remains inconclusive, as adjustments to the model fit experimental data regardless of the specific physical mechanism introduced.

An alternative line of research replaces the uniform flow assumption with a sheared velocity profile. While several studies have employed one-dimensional shear flow models [5, 7, 8], results indicate that incorporating sheared

***Corresponding author**: *lucas.bonomo@lva.ufsc.br.*

flow alone does not fully resolve the impedance mismatch [9]. A recent parametric study [8] found that shear effects become more significant at higher frequencies and larger Helmholtz numbers but remain a reasonable approximation for small ducts at low frequencies.

Beyond one-dimensional shear models, some studies have explored the impact of fully three-dimensional flow. Numerical experiments comparing solutions from the Convected Helmholtz Equation (CHE) with those from the Linearised Euler Equations (LEE) [10] indicate that assuming uniform flow introduces a bias error. This bias appears to be a significant contributor to the upstream-downstream impedance mismatch observed experimentally. Furthermore, when impedance eduction is performed using wavenumbers derived from a more physically accurate flow representation, the mismatch is significantly reduced, especially at higher frequencies [11].

A common assumption in the literature is that the specific formulation of the boundary layer profile does not significantly influence acoustic propagation, provided that the displacement thickness remains unchanged. However, early investigations [12] suggest that while this holds for downstream propagation, upstream propagation is sensitive to the boundary layer shape. Despite this, the influence of boundary layer shape on impedance eduction remains largely unexplored.

In this work, we propose a numerical experiment to evaluate how different boundary layer formulations affect impedance eduction. We use the Pridmore–Brown equation [13] to obtain axial wavenumbers for a sheared mean flow and apply these wavenumbers in a traditional impedance eduction routine [14] assuming uniform flow. By comparing realistic turbulent boundary layer profiles with simplified formulations commonly used in the literature, we assess the extent to which small variations in wavenumbers influence the educed impedance. Our findings aim to provide further insight into the limitations of current eduction methods and their sensitivity to flow representation.

## 2. GOVERNING EQUATIONS

For this study, we consider the infinite 2D duct depicted in Figure 1. The duct has width $W$ and an axial flow $\mathbf{u}_0 = U_0(x)\hat{e_z}$, assumed invariant in the axial direction. The wall at $x = -W/2$ has a locally reacting impedance $Z(\omega)$, while the other wall is rigid.

Acoustic propagation of a wave with $\exp(\mathrm{i}\omega t)$ time-dependence follows the Pridmore–Brown Equation (PBE)

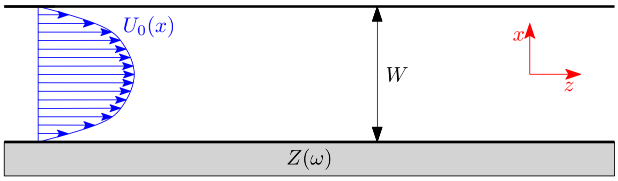


**Figure 1**. Schematic duct and coordinate system.

[13], given by

$$(\mathrm{i}\omega + \mathbf{u}_0 \cdot \nabla)\left(\frac{1}{c_0^2}(\mathrm{i}\omega + \mathbf{u}_0 \cdot \nabla)^2 \tilde{p}' - \nabla^2 \tilde{p}'\right) + 2\frac{\partial}{\partial z}(\nabla \tilde{p}' \cdot \nabla U_0) = 0. \quad (1)$$

Assuming an axial modal solution of the form $\tilde{p}'(x,z) = \tilde{p}'(x)\exp(-\mathrm{i}k_z z)$, with $k_z$ the axial wavenumber, we obtain

$$0 = \left(\nabla_\perp^2 + \frac{\omega^2}{c_0^2} - k_z^2\right)\tilde{p}' - k_z\left(\frac{U_0}{\omega}\nabla_\perp^2 - \frac{2}{\omega}\nabla_\perp U_0 \cdot \nabla_\perp + \frac{3\omega U_0}{c_0^2}\right)\tilde{p}'. \quad (2)$$

For boundary conditions, at the rigid wall $x = W/2$, the normal acoustic velocity vanishes, while at $x = -W/2$, the impedance boundary condition is

$$\frac{\mathrm{d}\tilde{p}'}{\mathrm{d}x} = -\frac{\mathrm{i}\omega}{Z}\tilde{p}'. \quad (3)$$

For a uniform flow, the PBE simplifies to the Convected Helmholtz Equation (CHE), and the impedance boundary condition reduces to the Ingard–Myers condition [15]. The problem is discretised using Chebyshev polynomials [16] and solved as a generalised eigenvalue problem.

## 3. VELOCITIES PROFILE SHAPE FUNCTIONS

The simplest formulation considered in this work is the sinusoidal flow profile, as presented by Ref. [17]. In this case,

$$\frac{U_0(x)}{c_0} = \begin{cases} M_s \sin\left(\dfrac{\pi\xi}{2\delta_s}\right), & 0 \le \xi \le \delta_s \\ M_s, & \xi > \delta_s, \end{cases} \quad (4)$$

where $M_s$ is the free-stream Mach number, and $\delta_s$ is the boundary layer thickness.

A commonly employed formulation nowadays is the hyperbolic tangent profile introduced by Ref. [18], given by

$$\frac{U_0(r)}{c_0} = M_c \left[ \tanh\left(\frac{1-r}{\delta_t}\right) + (1 - \tanh(1/\delta_t)) \times \left(\frac{1+\tanh(1/\delta_t)}{\delta_t} r + (1+r)\right)(1-r)\right], \quad (5)$$

where $M_c$ is the centreline Mach number, $r$ is the radial position, and $\delta_t$ is a shape factor. In this work, we use the coordinate transformation $r = 2|x|/W$ to obtain the flow profiles in the $x$ coordinate system.

For a realistic boundary layer velocity profile, we employ the universal wall law [19]

$$U^+ = \Pi + \int_0^{y^+} \frac{2}{1+\sqrt{1+4\kappa^2 {y^+}^2(1-\exp(-y^+/A^+))^2}} \mathrm{d}y^+, \quad (6)$$

where $\Pi$ is given by

$$\int_0^{y^+_{\max}} \frac{2}{1+\sqrt{1+4\kappa^2 {y^+}^2(1-\exp(-y^+/A^+))^2}} \mathrm{d}y^+ \times (y^+_{\max} - y^+)\left(\frac{y^+}{y^+\max}\right)^2 = \Pi. \quad (7)$$

where $y^+_{\max,x} = W u_\tau/\nu/2$. This formulation ensures continuity in the velocity profile derivative at the duct centreline.

## 4. MATERIAL AND METHODS

In this work, we consider a small rectangular duct representative of traditional liner impedance eduction facilities. The initial dimensions follow the Liner Impedance Test Rig from UFSC (LITR/UFSC), with a width of $W = 40\,\mathrm{mm}$, and a parametric study on the effect of duct width is planned.

For the lined wall impedance, we adopt an SDOF-like impedance given by

$$Z_{\mathrm{SDOF}}(\omega) = 2 - \mathrm{i}\left(\cot(k_0 h) - (0.03 k_0)^2\right), \quad (8)$$

with $h = 35\,\mathrm{mm}$, $k_0 = \omega/c_0$ is the free-field wavenumber, with $\omega$ the frequency in $\mathrm{rad\,s^{-1}}$ and $c_0$ the speed of sound in $\mathrm{m\,s^{-1}}$, which is representative of a typical liner in the considered frequency range and Mach number. The frequency range considered is 500–3000 Hz. The reference impedance is shown in Figure 2.

Velocity profiles are examined using three formulations presented: the turbulent universal wall law, Eqn. (6), a hyperbolic tangent profile, Eqn. (5), and a sinusoidal profile, Eqn. (4). The wall law parameters are fitted to experimental data from LITR/UFSC, yielding an average Mach number of $M = 0.279$, boundary layer thickness $\delta_{99\,\%} = 15.72\,\mathrm{mm}$, and displacement thickness $\delta^* = 1.70\,\mathrm{mm}$.

To assess the impact of velocity profiles on impedance eduction, we fit the hyperbolic tangent and sinusoidal profiles to match either the same bulk Mach number and boundary layer thickness or the same bulk Mach number and displacement thickness. The considered flow profiles are depicted in Figure 3

The number of points in the computational domain used for the pseudospectral solver was determined based on the critical case, which, for this study, corresponds to the universal wall law due to its high gradient near the walls.

For the numerical experiment eduction, we consider the traditional straightforward wavenumber based impedance eduction first proposed by Ref. [14]. Applying the Ingard–Myers boundary condition on the lined wall the CHE solution, it leads to the eigenvalue problem

$$k_x \tan(k_x W) - \frac{1}{\mathrm{i}k_0 Z}(\mathrm{i}k_0 - \mathrm{i}M k_z)^2 = 0, \quad (9)$$

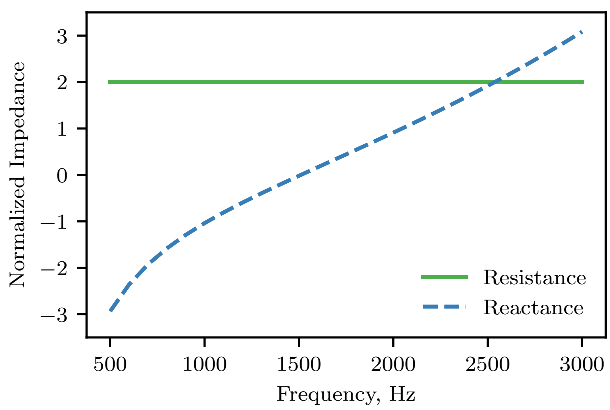


**Figure 2**. Reference impedance for the numerical experiments.

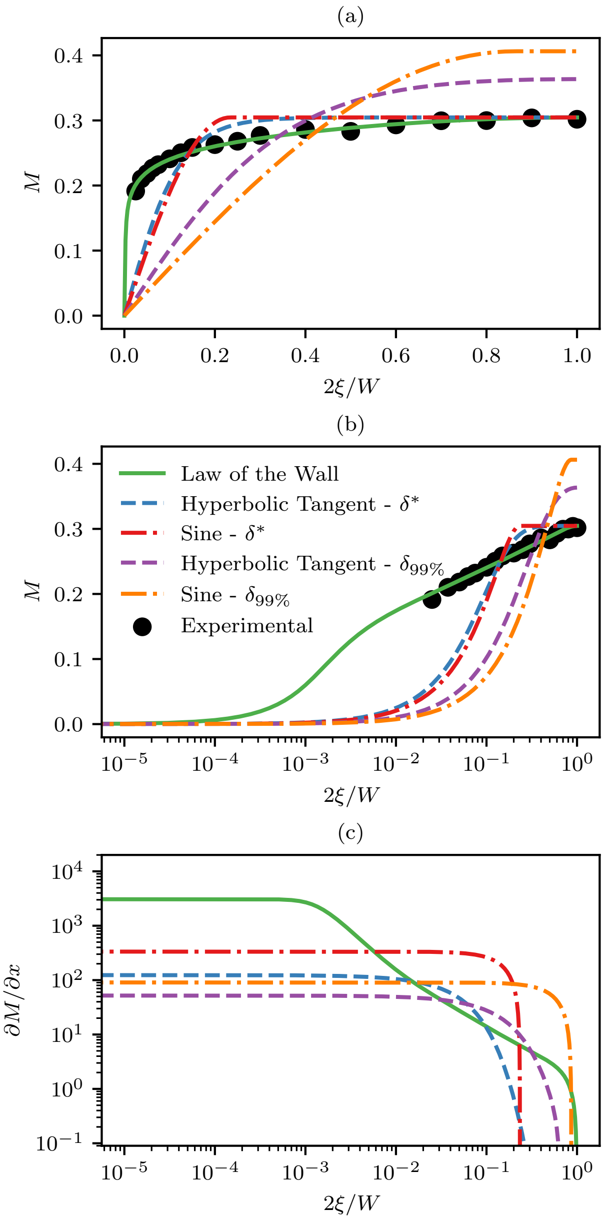


**Figure 3**. Flow velocities profiles in linear (a) and logarithmic (b) and velocities gradient (c) considered in the first step of this work.

where $k_x$ is the transverse wavenumber, and the dispersion relation given by

$$k_x^2 = (k_0 - Mk_z)^2 - k_z^2. \tag{10}$$

Once the axial wavenumber is known, it is straightforward to calculate the liner impedance from Eqn. (9) and Eqn. (10).

## 5. RESULTS AND DISCUSSION

First, we examine the wavenumbers obtained from the Pridmore–Brown equation for different velocity profile shapes, all with the same boundary layer thickness, $\delta_{99\,\%}$, and average Mach number $M$. These results are compared with the wavenumbers derived from the Convected Helmholtz equation with the Ingard–Myers boundary condition, which models the refraction at the boundary layer considering the average Mach number. The wavenumbers for the least attenuated mode, for both upstream ($k_z^-$) and downstream ($k_z^+$) propagation (corresponding to upstream and downstream sources, respectively), are presented in Figure 4.

Results suggest good agreement between all considered velocity profiles and the predictions from the CHE-IMBC for the real component of the wavenumber in both propagation directions. However, for the imaginary component of the wavenumber, which corresponds to the attenuation rate, significant differences are observed between the wavenumbers obtained for the hyperbolic tangent and sinusoidal flow profiles, compared to those obtained for the more realistic distribution given by the universal wall law, particularly for upstream propagation. Good agreement is observed between the CHE-IMBC solution and both the wall law and PBE solutions. This suggests that the IMBC may provide a good approximation for the typical range of impedance eduction. A possible explanation is that, due to the high gradient near the wall, even though the velocity distribution extends nearly across the entire duct half-width, the region where refractive effects are significant is confined to a much thinner region, making the infinitely thin hypothesis of the IMBC reasonable

Next, we consider the case where the different velocity distributions are adjusted to match the average Mach number and the boundary layer displacement thickness, $\delta^*$, rather than the boundary layer thickness, $\delta_{99\,\%}$. This approach is expected to improve the agreement between the acoustic attenuation predicted by the different flow profiles, particularly for downstream propagation [12]. The wavenumbers for the least attenuated mode in both upstream and downstream propagation are shown in Figure 5.

As expected, a better agreement is observed for the

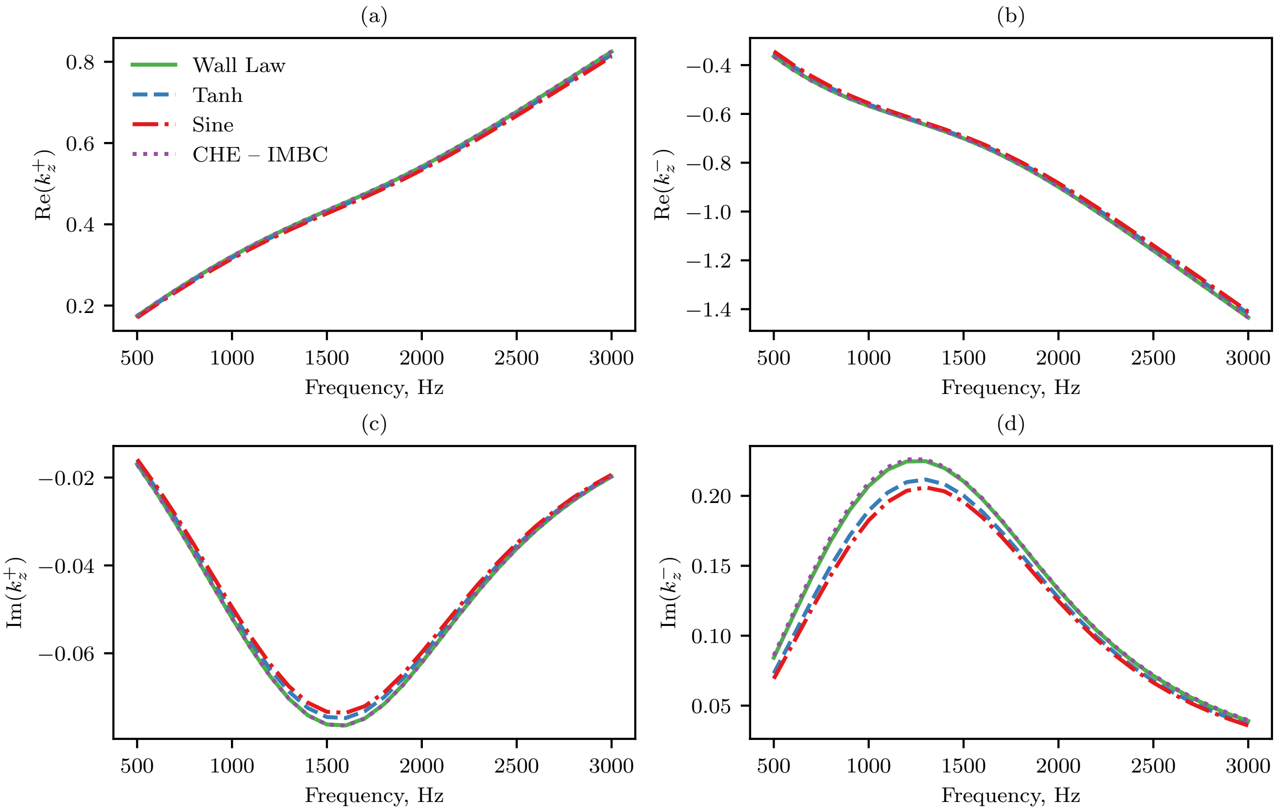


**Figure 4**. Wavenumbers obtained with different velocities distributions for the same $M$ and $\delta_{99.9\,\%}$.

wavenumbers obtained for the different velocity distributions, especially for downstream propagation. However, for upstream propagation, the wavenumbers for the hyperbolic tangent and sinusoidal flow profiles agree well with each other, but diverge from the solution for the wall law and the prediction for the CHE-IMBC. These initial results suggest that assuming a uniform flow and compensating for the refraction within the boundary layer using the Ingard–Myers boundary conditions provides better estimates of the acoustic field in impedance eduction facilities under typical test conditions, compared to solving for an explicit velocity distribution that is not representative of realistic conditions.

The next step, which is the main goal of this study, is to evaluate the impact of considering different flow velocity distribution shapes on the evaluation of the acoustic field in impedance eduction. We use the wavenumbers obtained for the least attenuated mode, considering the different velocity profile formulations, in the classical straightforward impedance eduction routine. This routine assumes uniform flow and the Ingard–Myers boundary condition to model the slip velocity at the wall. The impedances educed with the proposed numerical experiment, along with the wavenumbers obtained for the different velocity profiles in the PBE, are shown in Figure 6.

The impedances educed using the wavenumbers obtained from the exact solution of the hyperbolic tangent and sinusoidal flow profiles exhibit a similar trend regarding the mismatch observed experimentally between upstream and downstream acoustic sources (downstream and upstream propagation, respectively). At the lower frequency end, the upstream source case results in a lower resistance, with the opposite trend observed at higher frequencies. This behaviour is similar to what has been observed by Ref. [10]; however, in our case, the reference impedance is not the midpoint between the two curves. For the most realistic flow profile, the wall law, the conclusions differ significantly. At the lower frequency end,

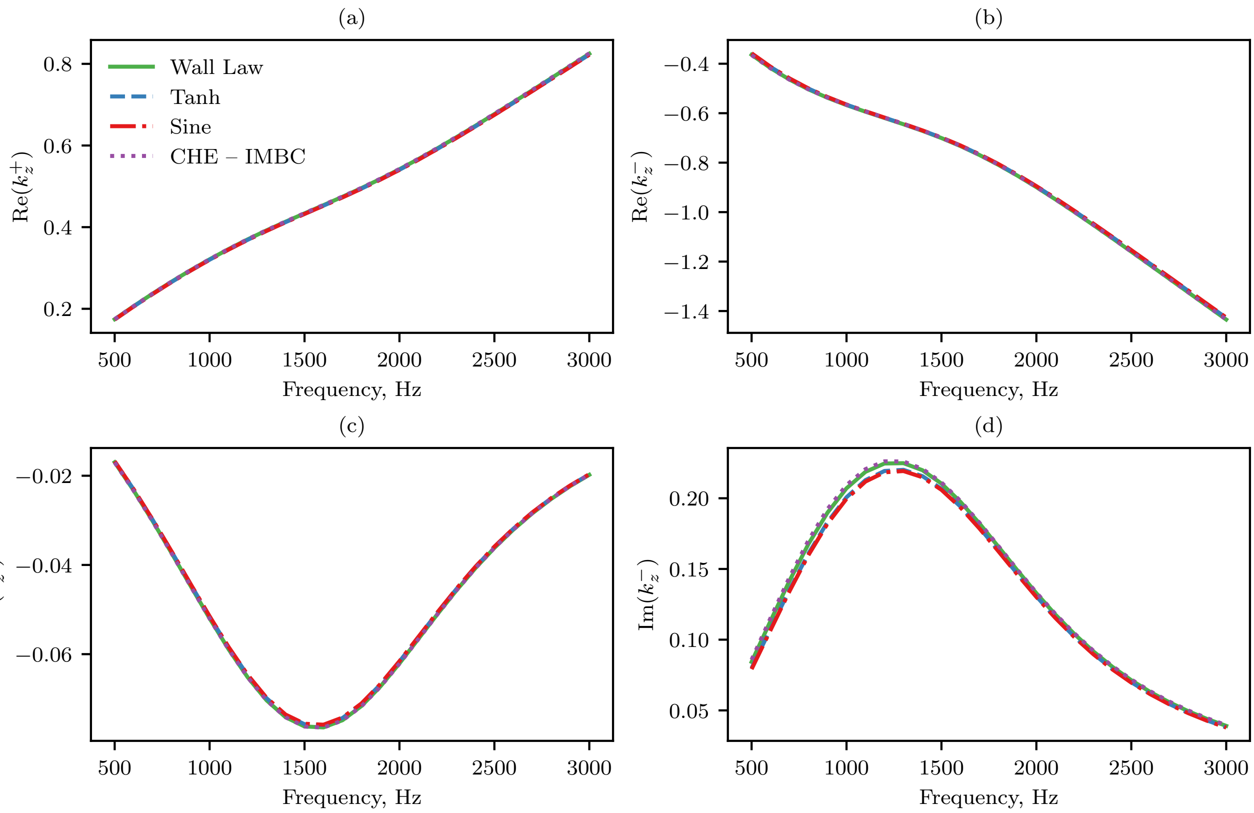


**Figure 5**. Wavenumbers obtained with different velocities distributions for the same $M$ and $\delta^*$.

the assumption of uniform flow with the IMBC introduces a small bias for both acoustic source positions, with good agreement observed between them. At higher frequencies, the curves diverge, with the upstream source (downstream propagation) surprisingly showing a greater deviation from the reference impedance.

## 6. CONCLUDING REMARKS

This work revisited the influence of sheared flow profiles on acoustic propagation in a 2D duct, extending Ref. [12] to impedance eduction. Comparing wavenumbers from the Pridmore–Brown equation with those from the Convected Helmholtz equation using the Ingard–Myers boundary condition (IMBC) showed that the IMBC performs well when the velocity profile is realistic, with improved agreement when matching the boundary layer displacement thickness.

A numerical experiment confirmed that IMBC-based eduction may introduce errors for non-realistic profiles but remains accurate for realistic flows. Iterative eduction with experimental data supported the findings of Ref. [5], showing reasonable agreement with the IMBC solution.

Overall, the IMBC is a reasonable simplification for low Mach number, small-duct impedance eduction in 2D ducts. Future work should extend this approach to realistic 3D ducts, as in Ref. [10], while preserving accurate flow profile representation.

## 7. ACKNOWLEDGMENTS

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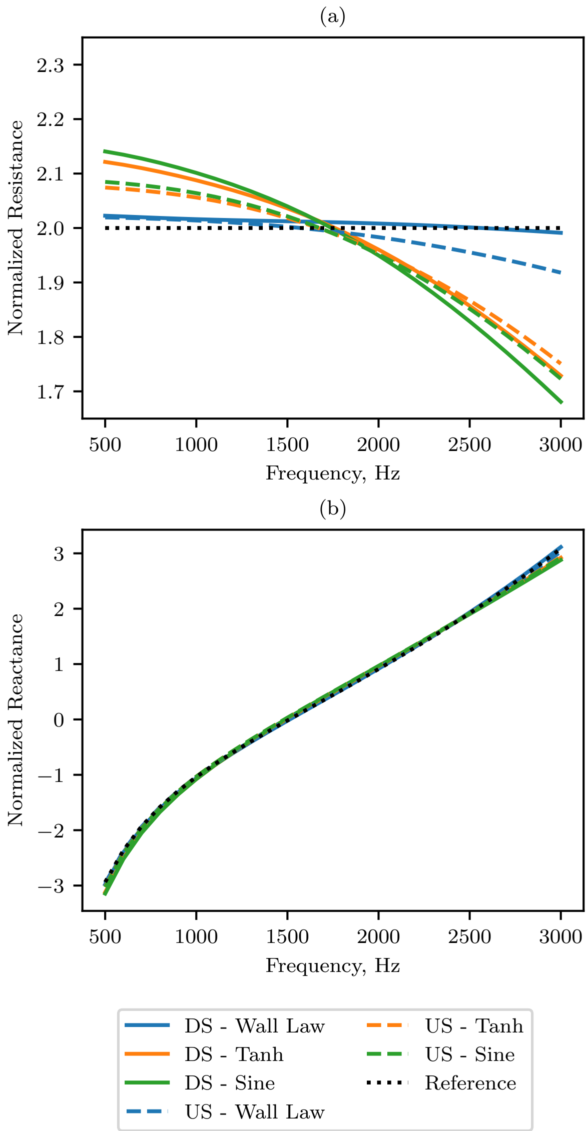


**Figure 6**. Educed impedances obtained for the wavenumbers evaluated for the PBE considering different velocities profile shapes with the same $\delta^*$. US - Upstream Source (downstream propagation), and; DS - Downstream Source (upstream propagation).

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