Comments on "linear stability analysis of a non-slipping mean flow in a 2D-straight lined duct with respect to modes type initial (instantaneous) perturbations", by Balint, Balint & Darau

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Abstract

This paper comments on a number of inaccuracies in the recently-published article by Balint, Balint and Darau (Applied Mathematical Modelling 35, pp. 1081–1095, 2011), concerning initial value acoustic perturbations to a steady mean nonslipping flow in a 2D duct: in particular, the neglect of antisymmetric solutions, and their comments on stability. Here, their dispersion relation is briefly rederived and simplified, two numerical results are presented demonstrating the existence of antisymmetric solutions in a specific case, and the inaccuracy in their comments on instability is highlighted.

Keywords: Acoustics; ducted sheared flow; impedance lining; stability.

We consider, as in [1], acoustic initial value perturbations to a steady mean nonslipping flow in a 2D duct. The fluid is a perfect gas, and the duct walls are acoustically lined. This situation has previously been extensively investigated [e.g. 2–5], and the following derivation follows these papers. In x, y Cartesian coordinates, taking the unit x-vector $\boldsymbol{e_x}$ along the centreline of the duct, the duct walls are at $y = \pm h$ and the mean flow has velocity $U_0(y)\boldsymbol{e_x}$ with constant density ρ_0 and pressure p_0. Acoustic perturbations are added to this mean flow, given by velocity perturbations $(\hat{u}(y)\boldsymbol{e_x} + \hat{v}(y)\boldsymbol{e_y})\exp\{\mathrm{i}\omega t - \mathrm{i}\alpha x\}$ and pressure perturbation $\hat{p}(y)\exp\{\mathrm{i}\omega t - \mathrm{i}\alpha x\}$. The Linearized Euler Equations in this case reduce to the Pridmore-Brown equation [6],

$$\hat{p}'' + \frac{2\alpha U_0'}{\omega - \alpha U_0}\hat{p}' + \big[(\omega - \alpha U_0)^2/c_0^2 - \alpha^2\big]\hat{p} = 0, \tag{1}$$

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where $'$ denotes $\mathrm{d}/\mathrm{d}y$ and c_0 is the speed of sound in the mean flow, with $\hat{u}$ and $\hat{v}$ given by

$$\hat{v} = \mathrm{i}\hat{p}'/\big(\rho_0(\omega - U_0\alpha)\big), \qquad \hat{u} = \frac{-\mathrm{i}}{\omega - U_0\alpha}\big[\mathrm{i}\alpha\hat{p}/\rho_0 - U_0'\hat{v}\big]. \tag{2}$$

Equation (1) is given as (2.2) in [1], with neighbouring equations equivalent to (2). The boundary conditions to be applied to (1) are that the impedance of the duct walls, $Z = \pm\hat{p}/\hat{v}$, is given:

$$Z\hat{p}' \pm \mathrm{i}\omega\rho_0\hat{p} = 0 \qquad \text{at} \qquad y = \pm h, \tag{3}$$

making use of no slip to give $U_0(\pm h) = 0$. This is (2.5) of [1]. The impedance Z is given using a mass–spring–damper model [7] as

$$Z = \rho_0 c_0\big(R + \mathrm{i}d\omega h/c_0 - \mathrm{i}bc_0/(h\omega)\big), \tag{4}$$

where d, b and R are the positive nondimensional mass, spring and damping coefficients. For the numerical examples given here and in [1], $d = 0.986$, $b = 1$ and $R = 2.6\times10^{-3}$ (an extremely small resistance in practice). The trivial solution to (1) subject to the boundary conditions (3) is $\hat{p} \equiv 0$. For special values of (ω, α) there exist nonzero solutions, which are termed modes; it is these modes that are the subject of [1].

The only case considered in detail in [1] is for a symmetric mean flow profile where $U_0(y) = U_0(-y)$. In this case, it can be shown that a nonzero mode $\hat{p}(y)$ is either even or odd; if $\hat{p}(0) \neq 0$ then $\hat{p}(y)$ is even and if $\hat{p}(0) = 0$ then $\hat{p}(y)$ is odd. In [1] it was proved in proposition 2.2 (with the proof in annex 2) that only even solutions exist. However, this proof is false, as (5) and (7) below give odd solutions, with an example odd solution plotted in figure 2*b* below. The fallacy comes in the assumption in annex 2 that $\hat{p}'(0) = 0$, which is only true for an even solution; the corresponding assumption for an odd solution would be $\hat{v}'(0) = 0$.

If $U_0(y) = \bar{U}_0$ for $y \in (-h+\delta, h-\delta)$, where $\bar{U}_0$ is a constant, then (1) reduces to the convected Helmholtz equation, with solution

$$\hat{p}(y) = C_1\cos(\mu y) + C_2\sin(\mu y) \qquad \text{where} \qquad \mu^2 = (\omega - \alpha\bar{U}_0)^2/c_0^2 - \alpha^2, \tag{5}$$

as given by Mariano [3, equation (7)] and Tester [4, equation (6a)]. Since [1] considered only even pressure perturbations they set $\hat{p}'(0) = 0$, and thus setting $C_2 = 0$ corresponds to their solutions. Within the sheared flow region $[-h, -h+\delta]$, let $\hat{p}_1$ be the solution of the initial value problem solving (1) subject (3) at $y = -h$ with $\hat{p}_1(-h) = 1$. Matching these two solutions in $[-h, -h+\delta]$ and $(-h+\delta, 0]$ with $C_2 = 0$ gives the even dispersion relation

$$\mu\tan\big(\mu(-h+\delta)\big) + \frac{\hat{p}_1'(-h+\delta)}{\hat{p}_1(-h+\delta)} = 0, \tag{6}$$

or equivalently,

$$\mu\tan\big(\mu(-h+\delta)\big) + \mathrm{i}\rho_0(\omega - \bar{U}_0\alpha)/Z_{\text{eff}} = 0, \qquad \text{where} \qquad Z_{\text{eff}} = \mathrm{i}\rho_0(\omega - \bar{U}_0\alpha)\frac{\hat{p}_1(-h+\delta)}{\hat{p}_1'(-h+\delta)},$$

This is the equivalent of both parts i) and ii) of proposition 2.3 of [1], as seen by taking $\lambda_1 = \mathrm{i}\mu$ and $\lambda_2 = -\mathrm{i}\mu$ in (6). It is also the equivalent of [4, equation (7)] and [5, equation (9)]. In order to solve this dispersion relation, numerical solutions are needed for $\hat{p}_1$; approximate solutions for small δ are given in [3, 4, 8–12], while an exact solution in a specific case is given by Goldstein and Rice [5]. An example of a solution to (6) is given in figure 1 and figure 2*a*.

The anti-symmetric solutions predicted not to exist and therefore missing from the analysis of [1] are given by taking $C_1 = 0$ in (5), giving the dispersion relation

$$\mu\cot\big(\mu(-h+\delta)\big) - \frac{\hat{p}_1'(-h+\delta)}{\hat{p}_1(-h+\delta)} = 0, \tag{7}$$

where we take $\mu\cot\big(\mu(-h+\delta)\big) = 1/(-h+\delta)$ for $\mu = 0$. Again this is the equivalent of [4, equation (7)] and [5, equation (9)], and again [3, 4, 8–12] are also relevant. An example of a solution of (7) is given in figure 2*b*.

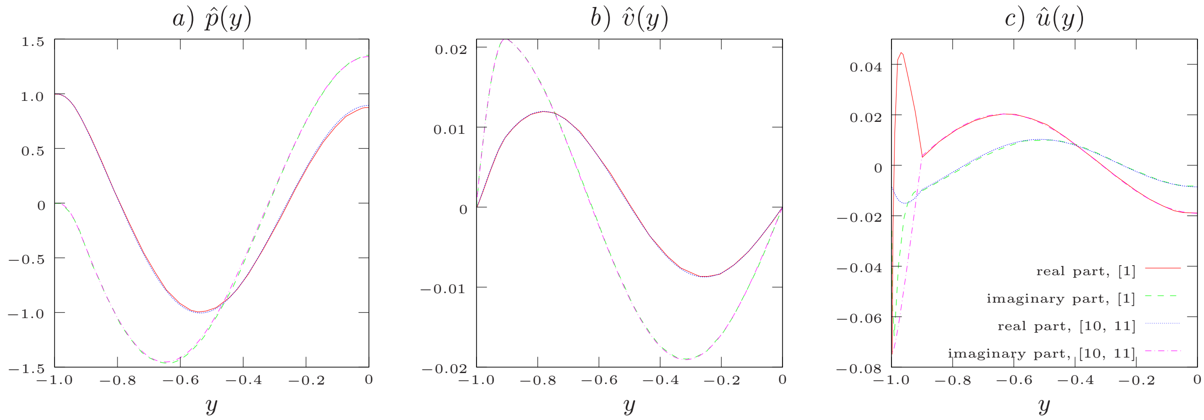

Figure 1: Equivalent of figure 2 of [1]. Mode with $\omega = -57 - 6.912\mathrm{i}\,\mathrm{s}^{-1}$, $\alpha = 0.5709 + 5.3453\mathrm{i}\,\mathrm{m}^{-1}$. $\rho_0 = 1.225\,\mathrm{kgm}^{-3}$, $c_0 = 340\,\mathrm{ms}^{-1}$, $h = 1\,\mathrm{m}$, $\delta = 0.1\,\mathrm{m}$, $\bar{U}_0 = 60\,\mathrm{ms}^{-1}$, $U_0(y) = -(y+h)(y+h-2\delta)\bar{U}_0/\delta^2$ for $y \in [-h, -h+\delta]$, Z is given by (4) with $d = 0.986$, $b = 1$ and $R = 2.6\times 10^{-3}$.

It should be noted here that the criticism in [1] of Tester [4] is misplaced. [1] state that "The applicability of the technique developed by Tester . . . is also questionable. That is because in the first step of this technique the pressure is uniform through boundary layer, i.e. its derivative with respect to y is equal to zero (also on the wall)." This is not true; it is the derivative with respect to $Y = y/\delta$ *to leading order* that is shown to be zero in [4]; i.e. the derivative with respect to y would occur at the *next order* in his asymptotic expansion and could be arbitrary. This is in line with both [1] and the asymptotic analyses of thin boundary layers in [8, 10–12].

Two numerical examples are presented here. Figure 1 shows the results from figure 2 of [1] compared with those of [10, 11] (adapted for Cartesian rather than cylindrical geometry). A good agreement is shown, apart from for $\hat{u}(y)$ within the sheared flow region $[-1, -0.9]$. If the formula for $\hat{u}(y)$ from (2) is replaced with

$$\hat{u} = \frac{-\mathrm{i}}{\omega - U_0\alpha}\left[\mathrm{i}\alpha\hat{p}/\rho_0 + U_0'\hat{v}\right],$$

then the results of [1] are reproduced; this shows that a sign error was made in the numerical calculation of $\hat{u}$ in [1], although fortunately this is irrelevant to the calculation of eigenvalues (ω, α). The pressure $\hat{p}(y)$ from figure 1 is shown across the whole duct in figure 2*a*. This is an even function, since that was the assumption made in [1], and the eigenvalue $\alpha = 0.5709 + 5.3453\mathrm{i}\,\mathrm{m}^{-1}$ is found using the dispersion relation for even solutions (6). Figure 2*b* shows an odd solution for $\hat{p}(y)$, with $\alpha = 0.6866 + 4.0777\mathrm{i}\,\mathrm{m}^{-1}$ given as a root of the dispersion relation for odd functions (7). This demonstrates that odd solutions do exist, countering proposition 2.2 of [1].

Finally we consider stability and instability with respect to these initial value modal perturbations. Here, by stability, we mean whether the zero solution is stable or unstable in time when perturbed by a bounded initial value modal solution. This implies that we should only consider $\alpha \in \mathbb{R}$, as otherwise the initial perturbation is not bounded. This is mentioned in [1, p. 1088], but the importance of this statement for stability is not sufficiently highlighted. To determine stability, we must determine the roots ω of the dispersion relation for every value of $\alpha \in \mathbb{R}$. Usually this step is accomplished numerically; in [1] there are numerical illustrations, but there is no calculation giving *all* ω roots for any $\alpha \in \mathbb{R}$. If, for a given $\alpha \in \mathbb{R}$, it is found that the only ω roots have $\mathrm{Im}(\omega) > 0$ then the zero solution of the linearized Euler equations is stable with respect to this initial value modal perturbation. But if for a given $\alpha \in \mathbb{R}$ it is found that there exists an ω root with $\mathrm{Im}(\omega) < 0$ then then the zero solution of the linearized Euler equations is not stable with respect to this initial value modal perturbation. Since in [1] the only numerical illustrations with $\mathrm{Im}(\omega) < 0$ where for $\mathrm{Im}(\alpha) \neq 0$, the statement [1, p. 1094] that "the existence of modes whose amplitude grows exponentially in time reveals linear instability" must be corrected; the numerical illustrations in [1] do not reveal linear instability, although they do not preclude it either. A conclusive study would proceed as indicated in this paragraph.

For details of an alternative stability analysis with respect to point source produced perturbations in sheared flow over an impedance lining in 2D, the reader is referred to Rienstra and Darau [12].

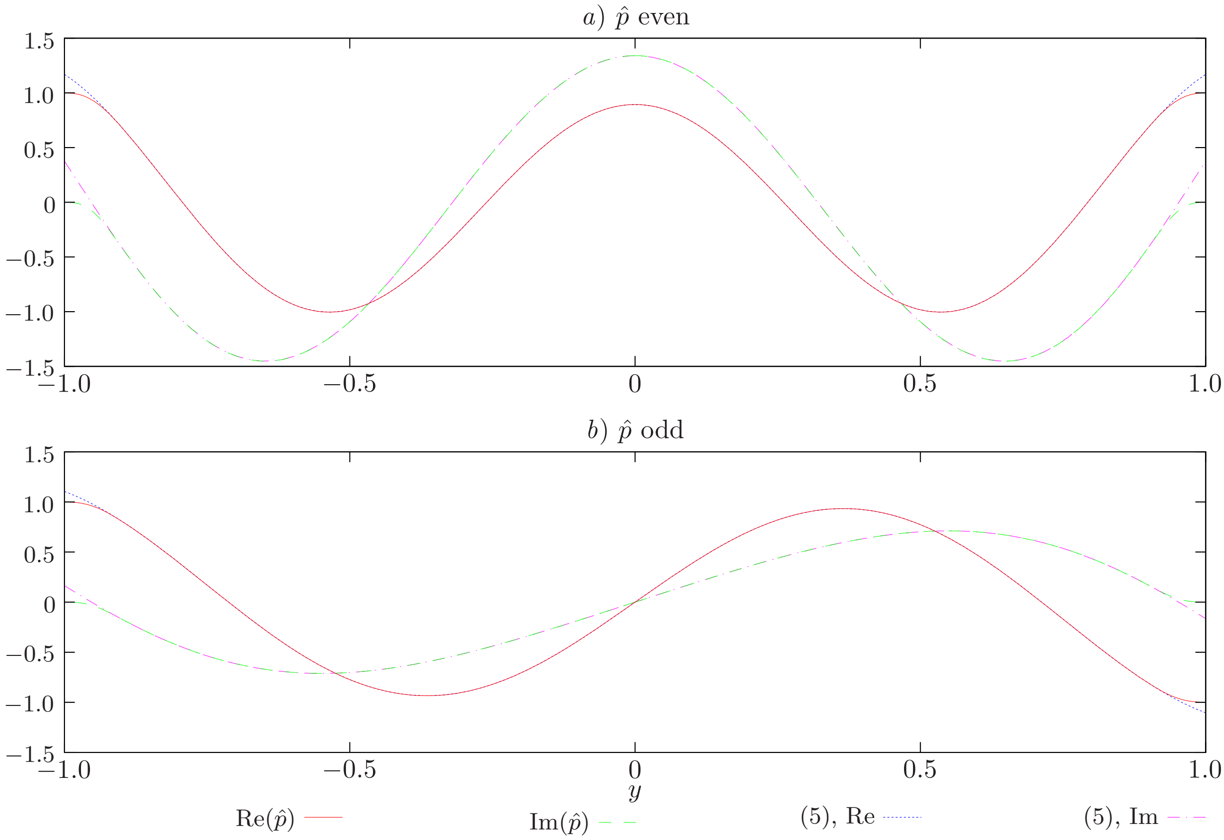

Figure 2: Examples of modes with even and odd $\hat{p}(y)$. Parameters are the same as for figure 1, with *a*) $\alpha = 0.5709 + 5.3453\mathrm{i}\,\mathrm{m}^{-1}$ and *b*) $\alpha = 0.6866 + 4.0777\mathrm{i}\,\mathrm{m}^{-1}$. Also shown are the uniform-flow solutions from (5).

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