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Models for Acoustically-Lined Turbofan Ducts

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Several models have recently been proposed for the impedance response of an acoustic lining; most recently, Rienstra's Enhanced Helmholtz Resonator model. All of these models have stability issues with nonzero mean flow, and there has been some debate over the existence of hydrodynamic instability waves over the surface of such acoustic linings. Mathematically, the standard proven Briggs–Bers stability analysis is not applicable. Computationally, the hydrodynamic modes are routinely ignored (in the frequency domain) and instabilities filtered out (in the time domain). This ambiguity causes significant problems for mode-matching and for scattering analysis.

The inadequacies of the recently-proposed "Crighton–Leppington" stability criterion are demonstrated through a number of counter-examples. It is then shown that any impedance lining model not capable of being analysed using the Briggs–Bers criterion is illposed, in the true mathematical sense, and an explanation is given as to why such models should not be used in practice.

Some commonly-used locally-reacting impedance boundaries (namely the mass–spring–damper, three-parameter, Helmholtz Resonator, and Enhanced Helmholtz Resonator models) are shown to be illposed, and an empirical impedance model is suggested that is wellposed and stable, whilst maintaining mass–spring–damper-like behaviour for the propagating and first few cutoff modes. It is suggested that this model may well resolve the ambiguity caused by surface modes in scattering and mode-matching problems.

A suggestion is also made for an additional fifth condition to be added to Rienstra's four conditions necessary for a locally-reacting impedance to be admissible.

I. Introduction

Consider flow in a duct with velocity $\boldsymbol{U} + \boldsymbol{\nabla}\phi\exp\{\mathrm{i}\omega t\}$, where $\boldsymbol{U}(\boldsymbol{x})$ is the mean flow and $\phi(\boldsymbol{x})$ is a small acoustic perturbation. The duct wall is impermeable to the mean flow, so that $\boldsymbol{U}\cdot\boldsymbol{n} = 0$ there, where $\boldsymbol{n}$ is the normal to the duct wall, oriented out of the fluid. For an acoustically lined duct wall, rather than modelling the physics of the lining, a simplified model of a linear and locally-reacting lining is usually assumed.[1–5] The response of the lining is then characterized by its impedance, $Z(\omega) = p/v$, where a time-harmonic pressure forcing $p\exp\{\mathrm{i}\omega t\}$ produces a time-harmonic fluid velocity $v\exp\{\mathrm{i}\omega t\}$ normal to the lining. With nonzero mean flow, the duct boundary is considered to be an infinitely-thin vortex sheet separating the moving fluid on one side and the fluid on the lining surface on the other. The effect of the lining is to relate the normal fluid velocity to the pressure on the lining side of the vortex sheet, so that the jump conditions across the vortex sheet then give the boundary condition (due to Myers[6])

$$\mathrm{i}\omega\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi = (\mathrm{i}\omega + \boldsymbol{U}\cdot\boldsymbol{\nabla} - (\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U})\cdot\boldsymbol{n})\,p/Z. \tag{1}$$

The question is now: what is a suitable model for $Z(\omega)$? Several models have been proposed. A simple one is the three-parameter model proposed by Tam & Auriault,[7] or equivalently the mass–spring–damper model proposed by Rienstra.[8] If the vortex-sheet displacement in the normal direction is given by $w(\boldsymbol{x}, t)$, then this model is

$$d\frac{\partial^2 w}{\partial t^2} + R\frac{\partial w}{\partial t} + bw = p, \qquad \Rightarrow \qquad Z(\omega) = R + \mathrm{i}d\omega - \mathrm{i}b/\omega, \tag{2}$$

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The justification of this model is that it captures three physical quantities, the inertia or mass d, the springiness b, and the damping R, and that these parameters may be set to give the correct behaviour for $Z(\omega)$ *locally* about some target frequency ω_0 and target impedance Z_0, possibly obtained from experimental measurement. This model is here referred to as the MSD model.

Another model is the Enhanced Helmholtz Resonator model (the EHR model) proposed by Rienstra,[2] which attempts to model the behaviour of a typical acoustic lining consisting of an array of Helmholtz resonators behind a perforated facing sheet. For this model

$$Z(\omega) = R + \mathrm{i}d\omega - \mathrm{i}\nu\cot(\omega L - \mathrm{i}\varepsilon/2), \tag{3}$$

where L is the depth of the Helmholtz resonators, $\nu > 0$ is a parameter scaling the cavity reactance, $\varepsilon > 0$ is a damping within the fluid in the cavity, and the speed of sound has been normalized to unity. Setting $\nu = 1$ and $\varepsilon = 0$ yields the original Helmholtz Resonator model (the HR model).

All of these models satisfy the four constraints set down by Rienstra[2] for a locally-reacting impedance to be admissible, which are that the impedance be what Rienstra called causal (two conditions, also interpretable as stability conditions), give real velocities for real pressures, and be energy absorbing. As a result of the analysis here, in §VI we propose that a fifth condition be added to these: that of wellposedness.

Rienstra demonstrated the existence of *surface modes* (modes localized close to the duct boundary) for sound in cylindrical ducts lined with a locally-reacting linear lining.[8] Using the MSD lining model, Rienstra suggested that one of these modes, present only with nonzero mean flow, might be interpreted as a downstream-growing instability, and termed this mode a hydrodynamic instability mode. More recently, Rienstra went on to suggest that such a mode should be included in scattering problems as a downstream-propagating instability, as this allowed a Kutta-like condition to be applied leading to smoother behaviour.[4] The standard Briggs–Bers stability analysis[9,10] (effectively a Fourier–Laplace transform method) turns out to be inapplicable in this case, and Rienstra described and then used an alternative stability analysis, which he termed the "Crighton–Leppington" criterion. This method, it is claimed, is based on the procedure used by Crighton & Leppington in Ref. 11. We give a brief review of both criteria in §II.

Crighton & Leppington[11] analysed the stability of a thin-plate shedding a vortex sheet, and the consequential Kelvin–Helmholtz instability. In their frequency-domain analysis, they discovered that by taking $\mathrm{Im}(\omega)$ sufficiently negative a pole lay on one side of their inversion contour, whereas for $\omega \in \mathbb{R}$ the pole lay on the other side. They argued that the former was the correct choice, since it gave a finite fluid velocity at the edge of the plate (a Kutta-like condition), although they admitted that "there is as yet no 'inner' problem solved to which any outer solution could be matched, and consideration of the outer problem alone cannot lead to any preference". However, they proposed that "the criteria for a causal solution, in this and in similar problems, are that $\phi(\boldsymbol{x}, k)$[a] must be analytic in k in the upper half-plane, and that $\phi \to 0$ as $\mathrm{Im}(k) \to \infty$ with $\mathrm{Re}(k)$ fixed. In practice these criteria are most conveniently met by calculating the solution for imaginary $k\ldots$, with the solutions for other values of k then inferred by analytic continuation." This is a different criterion than the one proposed as the "Crighton–Leppington" criterion in Ref. 4, and unfortunately, the "Crighton–Leppington" criterion as described in Ref. 4 is not in general valid, as we demonstrate with a number of examples in §III. For their initial-value formulation, Crighton & Leppington abandoned their frequency-domain analysis and instead followed a method of Jones & Morgan[12] which yielded an explicit solution in terms of ultradistributions.

The problems mentioned above with the stability analysis of the surface waves also, it turns out, have knock-on effects with the stability of numerical simulations of the acoustics of lined ducts with flow in the time domain. To force their stability, such numerical simulations always include an artificial damping term to filter out the instability.[3,7,13] This is connected with illposedness. In §IV we show that any model for which the Briggs–Bers criterion is inapplicable is illposed, and demonstrate what this means from a practical (computational and analytical) perspective. Moreover, we show that a certain class of locally-reacting linear impedance boundary models, which include (2) and (3), are illposed. A wellposed empirical boundary model is proposed in §V, which is stable provided certain restrictions are met, yet which yields a very similar impedance to the MSD model for all propagating and the first few cutoff duct modes; these modes being the ones of practical importance.

[a]Crighton & Leppington used a time-dependence $\exp\{-\mathrm{i}kt\}$, whereas in this paper we use $\exp\{\mathrm{i}\omega t\}$, and so ω is the complex conjugate of k.

II. Stability criteria

It is common, when presented with a linear partial differential equation in space and time, to look for a solution in terms of modes. For example, consider the differential equation

$$\frac{\partial G}{\partial t} + 3\frac{\partial G}{\partial x} - \frac{\partial^2 G}{\partial x^2} - G = 0.$$

One solution to this is $G(x,t) = A\exp\{\mathrm{i}t - \mathrm{i}kx\}$ for $k = 3\mathrm{i}/2 - \mathrm{i}\sqrt{7/2+\mathrm{i}}$, where k has positive real and imaginary parts. It therefore appears as a wave with a phase velocity in the positive x-direction, and which grows exponentially in the positive x-direction. Does this correspond to an instability propagating in the positive x-direction, or to an evanescent mode decaying in the negative x-direction?

The issue of instability is very closely linked with that of causality, as we will see below. The Briggs–Bers method considers the response of the system to a disturbance at $x = 0$ switched on at $t = 0$. Enforcing that the response to this disturbance occurs only after the disturbance is generated, the side of the disturbance on which the wave modes occur give their direction of propagation; if the mode above occurs in $x > 0$ then it is an instability, whereas if in $x < 0$ then it is stable and exponentially decaying.

A. The Briggs–Bers criterion

The Briggs–Bers criterion[9, 10] is effectively a Fourier–Laplace transform method. Consider the linear partial differential equation

$$\mathit{\Delta}\left(\mathrm{i}\frac{\partial}{\partial x}, -\mathrm{i}\frac{\partial}{\partial t}\right) G(x,t) = 0, \tag{4}$$

where $\mathit{\Delta}$ is the differential operator. We wish to analyse the stability of this system at a frequency $\omega_f \in \mathbb{R}$. As described by Briggs, we introduce a harmonic point-forcing term $\delta(x)H(t)\exp\{\mathrm{i}\omega_f t\}$ to the right hand side of (4), and then require that $G \equiv 0$ for $t < 0$ in order to satisfy causality. In order to solve this, we consider the transformation

$$\tilde{G}(k,\omega) = \int_0^\infty \int_{-\infty}^\infty G(x,t)\exp\{\mathrm{i}kx - \mathrm{i}\omega t\}\,\mathrm{d}x\mathrm{d}t.$$

Note that this is a Fourier transform $x \to k$ and a Laplace transform $t \to \mathrm{i}\omega$, the usual Laplace transform variable being $s = \mathrm{i}\omega$. These transformations are only valid provided the integrals converge. For the t-integral, convergence requires that $\mathrm{Im}(\omega)$ is sufficiently negative (so that $\mathrm{Re}(s)$ is sufficiently large) that the integrand tends to zero as $t \to \infty$. For the x-integral, we require that $|G(x,t)| \to 0$ sufficiently fast as $|x| \to \infty$, where sufficiently fast means fast enough that we may differentiate the number of times required by the differential equation. Transforming the differential equation with the added forcing term then gives

$$\mathit{\Delta}(k,\omega)\tilde{G}(k,\omega) = \frac{-\mathrm{i}}{\omega - \omega_f},$$

where $\mathit{\Delta}(k,\omega)$ is just a polynomial in k and ω. Inverting this transform gives the solution

$$G(x,t) = \frac{1}{4\pi^2}\int_{\mathcal{C}_\omega}\int_{\mathcal{C}_k} \frac{-\mathrm{i}\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}}{(\omega-\omega_f)\mathit{\Delta}(k,\omega)}\,\mathrm{d}k\mathrm{d}\omega, \tag{5}$$

where $\mathcal{C}_\omega$ and $\mathcal{C}_k$ are the inversion contours. Since the k-integral is a Fourier inversion, the $\mathcal{C}_k$ contour is along the real-k-axis. For $x < 0$ the contour may be closed in the upper-half k-plane and Jordan's Lemma applied, showing that for $x < 0$ the solution is given as a sum of residues of poles (i.e. modes) in the upper-half k-plane, and similarly for $x > 0$ and modes in the lower-half k-plane. This ensures that $G(x,t)$ has suitable behaviour as $|x| \to \infty$, as required above, since all modes decay exponentially quickly as $|x| \to \infty$. For the ω-integral, which is a Laplace inversion contour, the inversion contour for $s = \mathrm{i}\omega$ must be taken to the right of all poles of the integrand, so that $\mathcal{C}_\omega$ must be taken below all poles of the integrand. This requirement is equivalent to the requirement of taking $\mathrm{Im}(\omega)$ sufficiently negative that the forward-transform integral converges. It is also equivalent to causality, since for $t < 0$ the inversion contour may be closed in the lower-half ω-plane and Jordan's Lemma applied, to give $G \equiv 0$ for $t < 0$.

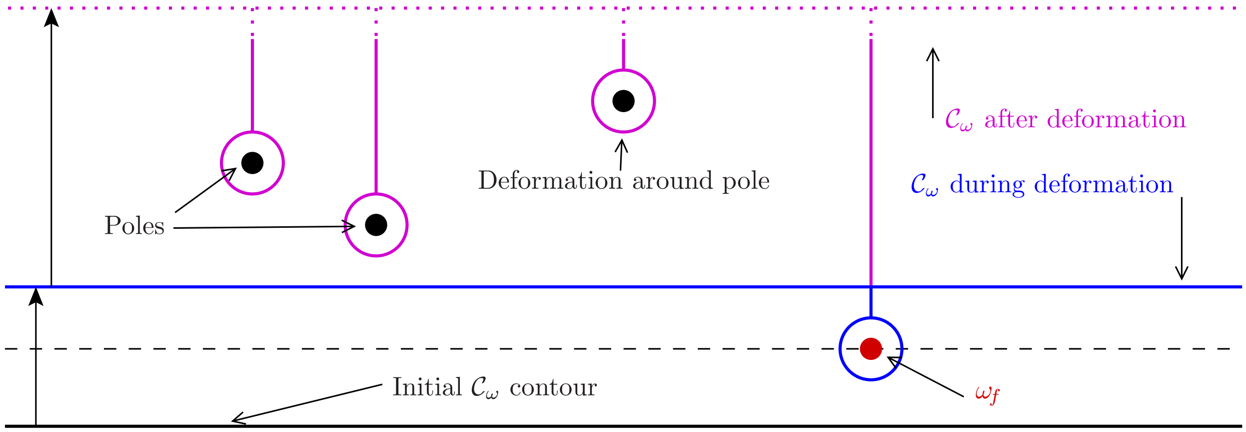

Figure 1. Illustration in the ω-plane of the Briggs–Bers contour-deformation process. The initial contour is transformed upwards, and around the poles to maintain analytic continuation.

We now assess the stability of the original differential equation, by looking at the long-time (or large-t) limit. Since the forcing term is time-harmonic, so too, it is hoped, will be the long-time limit, once all the transients from the sudden impulsive start have died away. The large-t limit is found by deforming the $\mathcal{C}_\omega$ contour into a steepest-descent contour. From (5), the steepest-descent lines are vertical lines, and so we deform the $\mathcal{C}_\omega$ contour upwards in the ω-plane, maintaining analyticity by deforming the contour around any poles of the integrand. The exponentially-dominating large-t contribution is then from the lowest pole in the ω-plane, with all other contributions from the remainder of the contour being exponentially small in comparison. This process is illustrated in figure 1.

During this process, the $\mathcal{C}_k$ contour may need to be deformed in order that no poles cross this contour in the k-plane, thereby maintaining the correct analytic continuation. In certain cases, two modes occurring on either side of the $\mathcal{C}_k$ contour may coincide as the $\mathcal{C}_\omega$ contour is deformed and *pinch* the $\mathcal{C}_k$ contour. Overall, there are three possibilities:

(a) The $\mathcal{C}_\omega$ contour may be deformed into the upper-half ω-plane, with the only deformation necessary being around the pole at ω_f on the real axis, and no deformation of the $\mathcal{C}_k$ contour being necessary. The dominant large-t contribution therefore comes from the pole at ω_f. The solution for $x < 0$ is hence a sum of modes of the form $\exp\{\mathrm{i}\omega_f t - \mathrm{i}kx\}$, where k satisfies the dispersion relation $\Delta(k, \omega_f) = 0$ and k lies in the upper-half k-plane. Similarly, the solution for $x > 0$ is the sum of poles occurring in the lower-half k-plane. Since all of these modes decay exponentially as $|x| \to \infty$, the system is *stable*.

If there are poles on the real-k-axis for ω_f, say $\Delta(k_0, \omega_f) = 0$ with $k_0 \in \mathbb{R}$, care must be taken to assign these to the correct side of the point forcing. For $\omega = \omega_f - \mathrm{i}\varepsilon$, we have $k = k_0 - \mathrm{i}\varepsilon/(\mathrm{d}\omega/\mathrm{d}k) + O(\varepsilon^2)$, so that if the group velocity $\mathrm{d}\omega/\mathrm{d}k$ is positive the mode will have originated from below the $\mathcal{C}_k$ contour and will therefore be present for $x > 0$, whereas if the group velocity is negative the mode will be present for $x < 0$. In both of these cases the system is referred to as being *neutrally stable*, and modes with $k \in \mathbb{R}$ are called *propagating*.

(b) If the $\mathcal{C}_\omega$ contour deformation into the upper-half k-plane (and around the pole at ω_f) is possible, but the $\mathcal{C}_k$ contour needed to be deformed to do so because of poles in the k-plane crossing the real-k-axis, the large-t solution is still a sum of modes of the form $\exp\{\mathrm{i}\omega_f t - \mathrm{i}kx\}$, as in (a), with k being solutions of $\Delta(k, \omega_f)$ and modes below the $\mathcal{C}_k$ contour occurring for $x > 0$ and modes above for $x < 0$. However, it is now possible that a mode that started in one half of the k-plane finishes in the other, and therefore grows exponentially in space as $|x| \to \infty$. In this case, the system is referred to as being *convectively unstable*.

(c) Finally, if two modes in the k-plane from opposite sides of the $\mathcal{C}_k$ contour coincide and pinch the $\mathcal{C}_k$ contour, say when $\omega = \omega_p$ and $k = k_p$, the $\mathcal{C}_\omega$ contour cannot be deformed further through this point while maintaining analytic continuity. In fact, locally about this point, $\Delta(k_p + \delta k, \omega_p + \delta\omega) = a\delta\omega + b\delta k^2 + O(\delta\omega^2, \delta k^3)$, so that the k-poles locally about $\omega = \omega_p + \delta\omega$ are given by $k = k_p \pm \sqrt{-a\delta\omega/b}$, and a branch cut in the ω-plane is necessary. Taking this branch cut to be along $\omega = \omega_p + \mathrm{i}y$ for $y > 0$, the $\mathcal{C}_\omega$ contour must be deformed around the singularity at $\omega = \omega_p$ and along the branch cut. The large-t contribution from this part of the integral turns out to be of the form $\exp\{\mathrm{i}\omega_p t - \mathrm{i}k_p x\}/\sqrt{t}$ both for $x < 0$ and $x > 0$, and since $\mathrm{Im}(\omega_p) < 0$, it is this that dominates the large-t solution, rather than the pole at ω_f. The system therefore chooses its own frequency at which to be unstable, and the solution grows exponentially in t for any position x; this instability is referred to as *absolute instability.*

This forms the Briggs–Bers stability criterion. First, we look for any values of ω with $\mathrm{Im}(\omega) < 0$ leading to double roots in the k-plane, and investigate whether these involve the collision of modes from opposite sides of $\mathcal{C}_k$. If so, the system is absolutely unstable, irrespective of the frequency we attempt to force it at. Otherwise, the system may be stable or convectively unstable, with the long-time solution being time-harmonic at the forcing frequency ω_f. The direction of propagation of modes may be found by tracking their location as ω is varied as $\omega = \omega_f + \mathrm{i}y$, with y starting suitably negative (we require $y < \mathrm{Im}(\omega)$ for any solution for ω of $\mathit{\Delta}(k, \omega) = 0$ for $k \in \mathbb{R}$, as this means we start below the original $\mathcal{C}_\omega$ contour). The direction of propagation of a mode $k(\omega_f)$ is given by which side of the real-k-axis it starts on, so that modes that start on one-side and end on the other as y goes from suitably negative to zero correspond to convective instabilities. We will look at some examples of how this criterion is applied in §III.

For the Briggs–Bers method to be valid, we required that the initial $\mathcal{C}_\omega$ contour be chosen below all values of ω for which $\mathit{\Delta}(k, \omega) = 0$ for any value of $k \in \mathbb{R}$, or looked at another way, we require the Laplace transform be defined for at least some values of s. However, in some problems, such as for the Kelvin–Helmholtz instability of a vortex sheet, $\mathrm{Im}(\omega)$ is not bounded below, and so it is not possible to choose such an inversion contour. Such problems are illposed, as we will investigate in §IV. It is for such problems that the "Crighton–Leppington" criterion has been proposed.

B. The "Crighton–Leppington" criterion

We use here the description of the "Crighton–Leppington" criterion given in Ref. 4. (Note that the description of the Briggs–Bers criterion in Ref. 4 is overly simplified and does not allow the same range of behaviours as the description given above.) For this "Crighton–Leppington" criterion, the motion of poles is traced in the k-plane as ω varies with $|\omega|$ fixed and $\arg(\omega)$ running from $-\pi/2$ to 0. If the frequency is negative, the deformation is for $\arg(\omega)$ running from $-\pi/2$ to $-\pi$ (Rienstra, private communication). Modes are counted right-running if they originate in the lower-half k-plane for ω purely imaginary, and left-running if they originate in the upper-half k-plane.

Ref. 4 has the proviso that $|\omega|$ must be "large enough" for this procedure to be applicable, although no justification of how large "large enough" is is given, nor an explanation of how to analyse stability for $|\omega|$ smaller than this. In fact, "large enough" means taking $|\omega|$ larger than $-\mathrm{Im}(\omega)$ for any zero of $\mathit{\Delta}(\omega, k)$ for $k \in \mathbb{R}$; in other words, so that the value of ω considered starts below the Briggs–Bers $\mathcal{C}_\omega$ contour. However, since the "Crighton–Leppington" criterion is designed for use when there is no bound to $-\mathrm{Im}(\omega)$, so that the Briggs–Bers criterion is inapplicable, it would seem that no value of $|\omega|$ is "large enough" in such cases.

III. Model stability examples

We now look at some examples of stability for some moderately-simple systems, chosen to allow a simple mathematical analysis while still including the relevant physics. Example 1 is analysed in detail, whereas the details for the other examples are omitted and only statements of their interesting features given.

A. Example 1

This example is of a convected wave-like equation with diffusive and self-exciting terms. The constants U, λ, and μ represent the convection, self-excitation and damping respectively, with the wave speed normalized to unity. The full differential equation is

$$\frac{\partial^2 G}{\partial x^2} - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 G + \lambda^2 G + \mu\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2 G}{\partial x^2} = 0, \tag{6}$$

giving

$$\mathit{\Delta}(k, \omega) = (\omega - Uk)^2 - k^2 + \lambda^2 - \mathrm{i}\mu(\omega - Uk)k^2.$$

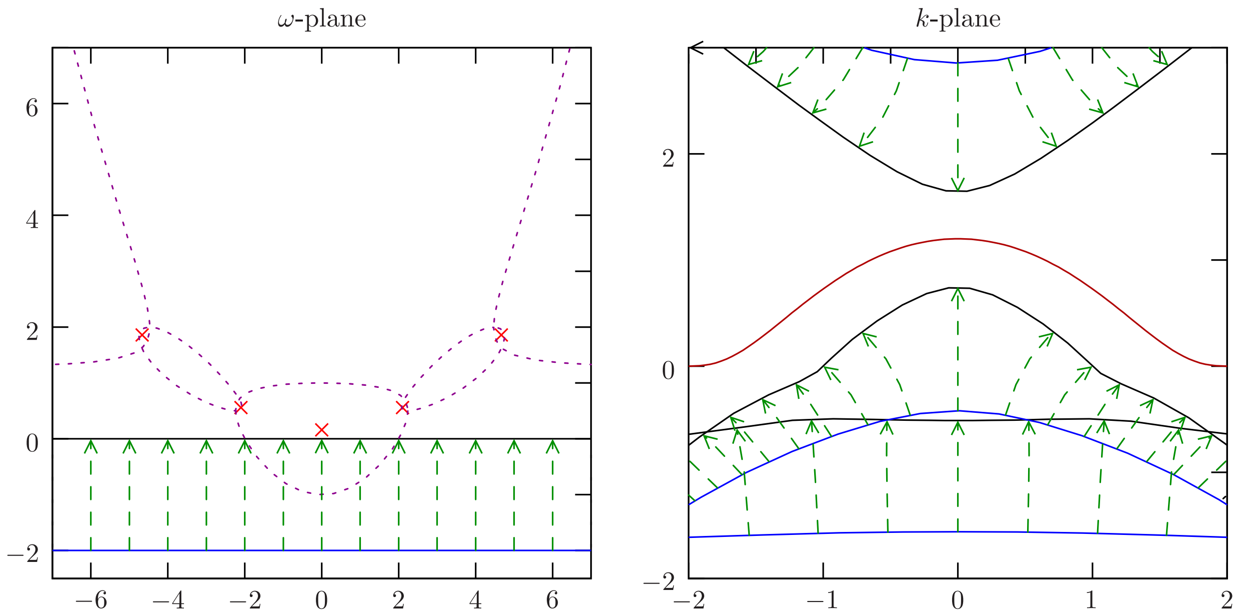

Figure 2. **$U = 2$, $\lambda = 1$, $\mu = 0.8$. The short-dashed lines are $\omega(k)$ for real k. Crosses denote values of ω from (7) giving a double root of $k(\omega)$. The blue curve shows the initial $\mathcal{C}_\omega$ contour and $k(\omega)$ on this contour. The black curve shows the final $\mathcal{C}_\omega$ contour and $k(\omega)$ on this contour. The green dashed arrowed lines show the motion between the two. The red line is the final $\mathcal{C}_k$ contour.**

Note that this is quadratic in ω and cubic in k, so that for a given frequency ω there are three modes (zeros of Δ), whereas for a given wavenumber k there are two. Solving $\Delta(k,\omega) = 0$ for ω in terms of k gives the two modes

$$\omega_\pm(k) = Uk + \frac{\mathrm{i}}{2}\mu k^2 \pm \sqrt{k^2 - \lambda^2 - \frac{1}{4}\mu^2 k^4},$$

so that $\omega_\pm(k) = Uk + \mathrm{i}\mu k^2(1 \pm 1)/2 \mp \mathrm{i}/\mu + O(k^{-2})$ as $|k| \to \infty$. By inspection, $\mathrm{Im}(\omega_\pm(k)) > -\lambda$ for all $k \in \mathbb{R}$. A graph of $\omega_\pm(k)$ for $k \in \mathbb{R}$ is given by the short-dashed line in the ω-plane in figures 2 and 3.

In order to analyse the stability of (6) using the Briggs–Bers technique, we must first locate the double-roots in the k-plane that may potentially pinch the $\mathcal{C}_k$ contour and lead to absolute instability. These double roots are found by solving $\Delta = \partial\Delta/\partial k = 0$ simultaneously. Eliminating k gives the quintic equation for ω

$$2U\omega A^2 + 2(1 - U^2 + \mathrm{i}\omega\mu)AB - 3\mathrm{i}U\mu B^2 = 0 \tag{7}$$

$$A = 2(1-U^2)^2 + 4\mathrm{i}\omega\mu(1+2U^2) - 2\omega^2\mu^2 \qquad B = 7\mathrm{i}U\mu\omega^2 + 9\mathrm{i}U\mu\lambda^2 - 2U\omega(1-U^2).$$

The numerical solutions of (7) are shown as crosses in figures 2 and 3.

We now apply the Briggs–Bers procedure to (6) for $U = 2$, $\lambda = 1$, and $\mu = 0.8$, as shown in figure 2. Initially, we take the $\mathcal{C}_k$ contour along the real-k-axis and the $\mathcal{C}_\omega$ contour to be $\mathrm{Im}(\omega) = -2$. We then deform the $\mathcal{C}_\omega$ contour upwards onto the real axis, as shown by the dashed arrowed lines. In so doing, the poles in the k-plane move, and we must deform the $\mathcal{C}_k$ contour upwards in order to avoid poles crossing the contour. No double roots hinder us in this process, so that no absolute instabilities exist for these parameters. Note, however, that for $|\mathrm{Re}(\omega)| < 2$ poles in the lower-half k-plane have moved into the upper-half, and since these are below the $\mathcal{C}_k$ contour, they represent right-propagating waves which are therefore exponentially growing downstream. The system is therefore convectively unstable for real frequencies $|\omega| < 2$, and stable for $|\omega| \geq 2$. For $|\omega| = 2$ the system supports neutrally-stable (i.e. propagating) modes.

If instead we had used $U = 1.2$, the situation is significantly different, as shown in figure 3. There is now a double-root in the lower-half ω-plane that involves two poles in the k-plane from opposite sides of the $\mathcal{C}_k$ contour which converge and pinch the contour. The $\mathcal{C}_\omega$ contour must therefore be deformed around this pinch frequency as shown. The dominant large-t contribution comes from this frequency, rather than any imposed forcing frequency ω_f, showing that in this case the system is absolutely unstable with dominant frequency $\omega_p \approx -0.592\mathrm{i}$.

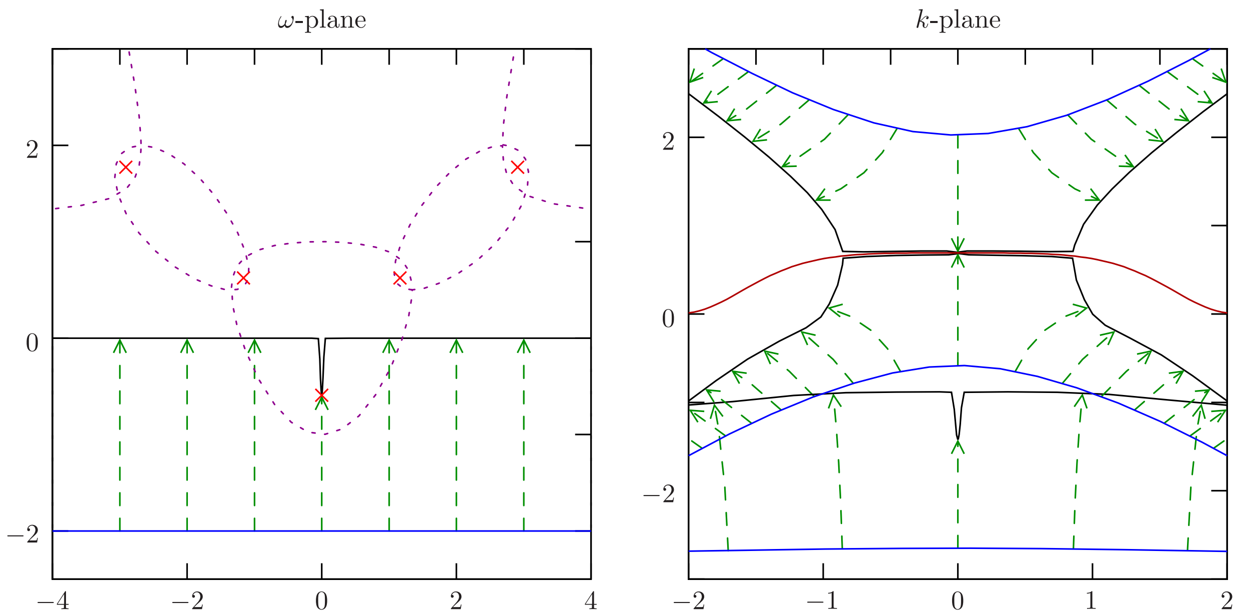

Figure 3. As for figure 2 but with $U = 1.2$.

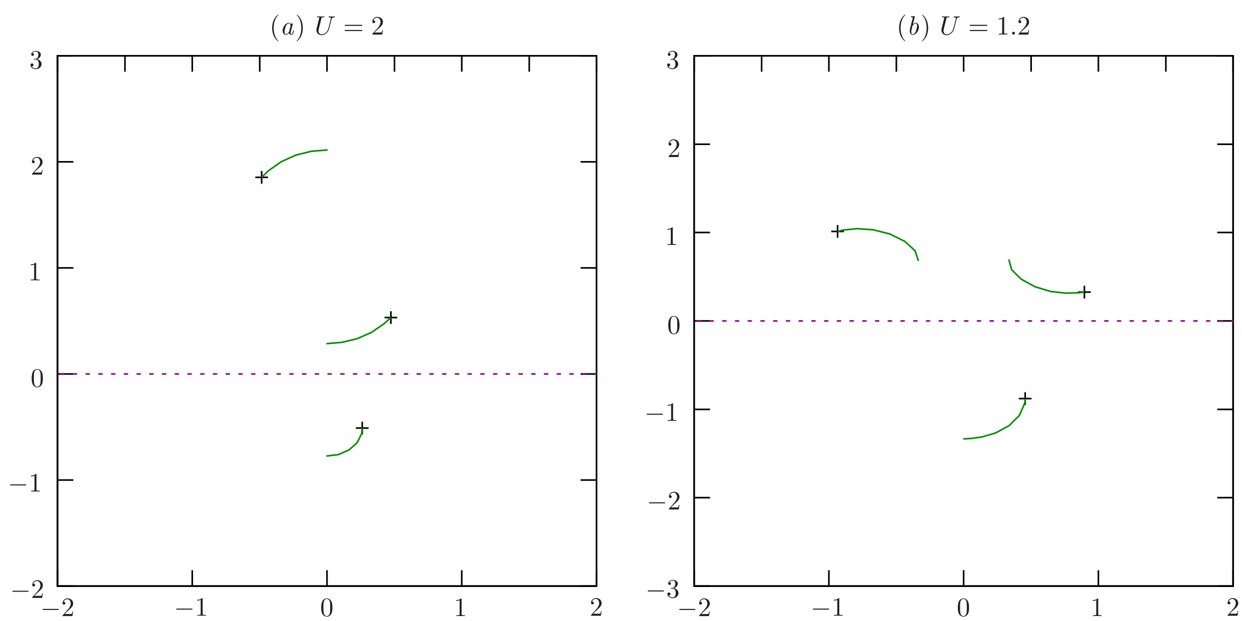

Figure 4. In the k-plane, modes $k(\omega)$ for $|\omega| = 0.5$ and $-\pi/2 < \arg(\omega) < 0$, with $\lambda = 1$, $\mu = 0.8$. Crosses denote $\omega = 0.5$.

If instead a "Crighton–Leppington" stability analysis had been applied, as shown in figure 4 for a frequency $\omega = 0.5$, our conclusions would have been different. In both cases considered, figure 4 shows no modes crossing from one half of the k-plane to the other, implying that no instabilities should be present in either case at a frequency of $\omega = 0.5$, and that in both cases the system should have two upstream-propagating modes and one downstream-propagating mode. However, in the first case $\omega = 0.5$ does have an unstable mode, and in fact has two downstream-propagating modes and one upstream-propagating mode, whereas in the second case the system is absolutely unstable, and it does not make sense to consider a time-harmonic solution with frequency $\omega = 0.5$. We therefore conclude that the "Crighton–Leppington" criteria is erroneous in this case.

B. Example 2

Another model problem very similar to example 1 is

$$c^2\frac{\partial^2 G}{\partial x^2} - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 G + \lambda^2\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)G + \mu\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2 G}{\partial x^2} = 0,$$

where the constants U, c, λ and μ represent the convection, wave speed (effectively the compressibility), self-excitation and damping terms respectively. If $c = 0$, this reduces to example 4, whereas otherwise it is very similar in its stability analysis to example 1.

For this example,

$$\varDelta(k,\omega) = (\omega - Uk)^2 - c^2k^2 + \mathrm{i}(\omega - Uk)\lambda^2 - \mathrm{i}\mu(\omega - Uk)k^2,$$

giving

$$\omega_\pm(k) = Uk + \frac{\mathrm{i}}{2}\mu k^2 - \frac{\mathrm{i}}{2}\lambda^2 \pm \sqrt{\left(\frac{1}{2}\mu\lambda^2 + c^2\right)k^2 - \frac{\lambda^4}{4} - \frac{1}{4}\mu^2k^4},$$

with double-k-roots occurring for values of ω satisfying the quintic equation

$$\begin{aligned}
U(2\omega + \mathrm{i}\lambda^2)A^2 + 2(c^2 - U^2 + \mathrm{i}\omega\mu)AB - 3\mathrm{i}U\mu B^2 &= 0\\
A = 2(c^2 - U^2)^2 - 6\mu U^2\lambda^2 + 4\mathrm{i}\omega\mu(c^2 + 2U^2) &- 2\omega^2\mu^2\\
B = 7\mathrm{i}U\mu\omega^2 - 2U\omega(c^2 - U^2 + 4\mu\lambda^2) &- \mathrm{i}U\lambda^2(c^2 - U^2).
\end{aligned}$$

Note that this solution, and therefore the stability analysis, must be computed numerically. Doing so once again demonstrates the inadequacy of the "Crighton–Leppington" criterion.

C. Example 3

This example is example 1 but with $\mu = 0$,

$$\frac{\partial^2 G}{\partial x^2} - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 G + \lambda^2 G = 0 \qquad \Rightarrow \qquad \varDelta(k,\omega) = (\omega - Uk)^2 - k^2 + \lambda^2.$$

It is included here as a separate example because the Briggs–Bers analysis may be done entirely analytically in this case. Solving $\varDelta(k,\omega) = 0$ for ω in terms of k, or for k in terms of ω, gives two roots,

$$\omega_\pm(k) = Uk \pm \sqrt{k^2 - \lambda^2}, \qquad k_\pm(\omega) = \frac{-U\omega}{1 - U^2} \pm \frac{1}{1 - U^2}\sqrt{\omega^2 + (1 - U^2)\lambda^2}.$$

In the subsonic case $0 < U < 1$ this system is absolutely unstable, with an obvious double root when $\omega = -\mathrm{i}\lambda\sqrt{1 - U^2}$ and $k_\pm = \mathrm{i}U\lambda/\sqrt{1 - U^2}$. In the supersonic case $U > 1$ the $\mathcal{C}_k$ contour sits above all poles in the k-plane, and therefore there is no possibility of absolute instability, since there are no poles above the contour to pinch it. All modes are therefore downstream-propagating. One of the two modes is a convective instability for $|\omega| < \lambda\sqrt{U^2 - 1}$. For $\lambda\sqrt{U^2 - 1} < |\omega| < \lambda U$ both modes are neutrally-stable (i.e. propagating), although one mode is anomalous in that it has an upstream-pointing group velocity. For $|\omega| > \lambda U$ the system supports two neutrally-stable (i.e. propagating) modes, with the group velocities of both pointing downstream. The "Crighton–Leppington" criterion incorrectly predicts the direction of propagation of one of the modes (and hence its stability) for $|\omega| < \lambda$, and of course also incorrectly predicts the subsonic case, as the "Crighton–Leppington" criterion does not consider absolute instability.

D. Example 4

Rather than a wave-like system as has been considered in the examples above, this example is of a self-exciting advection–diffusion type,

$$\frac{\partial G}{\partial t} + U\frac{\partial G}{\partial x} - \frac{\partial^2 G}{\partial x^2} - G = 0, \qquad \Rightarrow \qquad \mathit{\Delta}(k,\omega) = \mathrm{i}(\omega - Uk) + k^2 - 1$$

where U is the convection velocity. This gives

$$\omega = Uk - \mathrm{i}(1-k^2) \qquad \text{and} \qquad k_\pm = \mathrm{i}U/2 \pm \sqrt{1 - U^2/4 - \mathrm{i}\omega},$$

with an obvious double root for $\omega = \omega_p \equiv \mathrm{i}(U^2/4 - 1)$ and $k_\pm = k_p \equiv \mathrm{i}U/2$. This example has one upstream and one downstream mode for all values of U. For $0 < U < 2$ the system is absolutely unstable, since ω_p belongs to the lower-half ω-plane and the colliding modes are from opposite sides of the $\mathcal{C}_k$ contour. Otherwise, the downstream-propagating mode is convectively unstable for $|\omega| < U$, neutrally stable for $|\omega| = U$, and stable (and exponentially decaying) for $|\omega| > U$, while the upstream-propagating mode is always stable and exponentially decaying. The "Crighton–Leppington" criterion incorrectly predicts the downstream-propagating convectively-unstable mode to be a stable upstream-propagating mode for $|\omega| < 1$, and additionally fails to detect the absolute instability for $0 < U < 2$. Further details of this example are given in §2.3 of Ref. 14 (which, while qualitatively correct, has an incorrect factor of two for $k_\pm$; the solution given above is correct).

IV. Illposedness

As mentioned, the Briggs–Bers procedure is only applicable when the system being analysed has a bounded exponential growth rate, so that $\mathit{\Delta}(k,\omega) \neq 0$ for any $k \in \mathbb{R}$ for any ω with $\mathrm{Im}(\omega) < \eta$. However, for sound in a cylindrical duct with mean flow and an MSD lining model, we show in the next section that $\mathrm{Im}(\omega(k))$ is unbounded below, and so no such η exists. This was the argument for using the alternative "Crighton–Leppington" criterion, which we have just seen is inadequate. In fact, any system for which $\mathrm{Im}(\omega(k))$ is not bounded below is illposed. We now investigate this further.

Given an initial condition $G(x,0) = f(x)$, let us evolve this forward a time t to give $G(x,t) = (T_t\boldsymbol{f})(x)$, where the operator T_t, meaning evolve forward a time t, is a linear operator on functions $f(x)$. Specifically, T_t should be a continuous linear operator on the complete normed space (Banach space) of functions $\boldsymbol{f} \in X$, which could, for example, be the space of all uniformly-continuous functions under the sup-norm. Then the operator T_t should satisfy

$$1)\quad T_0 = I, \qquad 2)\quad T_{t+s} = T_tT_s \quad \forall t,s \geq 0, \qquad 3)\quad \lim_{t\to 0} T_t\boldsymbol{f} = \boldsymbol{f} \quad \forall \boldsymbol{f} \in X.$$

Yosida (Ref. 15, pp. 232–233) showed that these conditions imply that

$$\|T_t\boldsymbol{f}\| \leq M\mathrm{e}^{\eta t}\|\boldsymbol{f}\|,$$

for some M and η, so that the system necessarily has a finite maximum exponential growth rate. Yosida (Ref. 15, pp. 240–241) went on to show that this implied that $\mathcal{L}(s)$ was well defined for all complex s with $\mathrm{Re}(s) > \eta$, where

$$\mathcal{L}(s)\boldsymbol{f} = \int_0^\infty \mathrm{e}^{-st}T_t\boldsymbol{f}\,\mathrm{d}t$$

is a Laplace transform in t. In our notation, this implies that $\mathit{\Delta}(k,\omega) \neq 0$ for $\mathrm{Im}(\omega) < -\eta$ and $k \in \mathbb{R}$.

Therefore, if $\mathrm{Im}(\omega(k))$ is not bounded below for $k \in \mathbb{R}$, the problem is illposed, in the sense that the operator T_t (meaning evolve forward our initials conditions by a time t) is not continuous at $t = 0$. This is, in effect, because arbitrarily quickly growing instabilities may be present that can grow arbitrarily large between time $t = 0$ and time $t = 0^+$.

What does this mean practically? For sound in a cylindrical duct with an MSD impedance boundary and mean flow, we show in the next section that there are surface modes for which $\omega \sim -\mathrm{i}Nk^{1/2}$ for a real constant N as $|k| \to \infty$, so that the arbitrarily quick exponential growth occurs at arbitrarily short wavelengths. When numerically simulating such a system, it is found that for fine meshes the numerics are unstable at the grid scale, with finer meshes becoming unstable more rapidly, and these instabilities are therefore routinely filtered out.[3,7,13] However, since this instability can now be seen as the numerics attempting to accurately simulate the underlying mathematical differential equation, which has no regular mathematical solution, once part of the numerical solution is filtered out there is no justification that what is left is of any relevance to the physical problem being modelled. We have seen above that the illposedness causes problems with stability analysis, and it also causes problems for mode-matching[1,5] and scattering.[4,16]

Illposedness is well known in the context of the Kelvin–Helmholtz instability of a vortex sheet,[11,12] for which solutions were given in terms of ultradistributions. This problem was resolved, at least in some regards, by Jones,[17] who regularized the problem by considering a shear layer of finite thickness h. For small but nonzero h an instability was present and expressible in terms of conventional functions, which in the limit $h \to 0$ yielded the previously discovered ultradistribution result.

What seems to be needed, therefore, is a regularization of the MSD impedance boundary model that is wellposed. Unfortunately, all the proposed locally-reacting impedance boundary models considered here share the illposedness of the MSD boundary model, as we show next.

A. The illposedness of a common class of impedance boundary models

Consider a uniform cylindrical duct, with centreline in the x-direction and cross-section described by polar coordinates r, θ. We nondimensionalize lengths by the duct radius and speeds by the speed of sound. The velocity of the fluid is given by $\boldsymbol{u} = U\boldsymbol{e_x} + \boldsymbol{\nabla}\phi$, where ϕ is the acoustic perturbation and U is the steady mean flow Mach number. The solution for ϕ is

$$\phi = \mathrm{J}_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \qquad \alpha^2 = (\omega - Uk)^2 - k^2,$$

were J_m is the mth Bessel function of the first kind. The boundary condition (1) becomes

$$\frac{\alpha \mathrm{J}_m'(\alpha)}{\mathrm{J}_m(\alpha)} - \frac{(\omega - Uk)^2}{\mathrm{i}\omega Z(\omega)} = 0,$$

where $Z(\omega)$ is the boundary impedance.

When considering the boundedness or otherwise of $\mathrm{Im}(\omega)$ for $k \in \mathbb{R}$, it is the surface modes that cause the potentially unbounded behaviour, with the other acoustic modes being well behaved. The surface modes are predicted using the asymptotic dispersion relation of Ref. 18 (a modification of the original asymptotic surface mode dispersion relation of Rienstra[8]),

$$\sqrt{k^2 + m^2 - (\omega - Uk)^2} - \frac{(\omega - Uk)^2}{\mathrm{i}\omega Z} = 0, \tag{8}$$

where it is required that $\mathrm{Re}(\sqrt{\cdots}) > 0$. Rearranging to eliminate the square root gives

$$\left(k^2 + m^2 - (\omega - Uk)^2\right)(\mathrm{i}\omega Z)^2 - (\omega - Uk)^4 = 0 \qquad \text{with } \mathrm{Re}\left(\frac{(\omega - Uk)^2}{\mathrm{i}\omega Z}\right) > 0 \tag{9}$$

We now consider the behaviour of ω as $k \to \pm\infty$ with k real. Assuming that $\mathrm{i}\omega Z = A\omega^\sigma$ to leading order, we have three cases depending on whether $\sigma > 1$, $\sigma = 1$, or $\sigma < 1$,

$$\begin{array}{lll} \sigma > 1 & \omega = Nk^{1/\sigma} & N^\sigma = \pm U^2/(A\sqrt{1-U^2}) \\ \sigma = 1 & \omega = Nk & \left((1-U^2) + 2UN - N^2\right)A^2N^2 - (N-U)^4 = 0 \\ \sigma < 1 & \omega = Uk + Nk^{(\sigma+1)/2} & N^2 = \pm AU^\sigma. \end{array}$$

For the MSD boundary model given in (2), $\mathrm{i}\omega Z = -d\omega^2 + O(\omega)$, so that there exists a surface mode with $\omega \sim -\mathrm{i}|k|^{1/2}U(d^2(1-U^2))^{-1/4}$, implying $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$. For the HR and EHR boundary models (3), since $-\mathrm{i}\cot(x + \mathrm{i}y) \to 1$ as $y \to -\infty$, we again have $\mathrm{i}\omega Z = -d\omega^2 + O(\omega)$ for both models, again giving $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$.

If d were set to zero in any of these models, we are then in a different regime where $\sigma = 1$. However, such models with $d = 0$ are incapable of modelling typical acoustic linings, for which $\mathrm{Im}(Z) > 0$ for sufficiently large ω (see, e.g., figure 4 of Ref. 19). Even if $d = 0$, it is not clear that the unbounded behaviour of $\mathrm{Im}(\omega)$ for $k \in \mathbb{R}$ would be avoided.

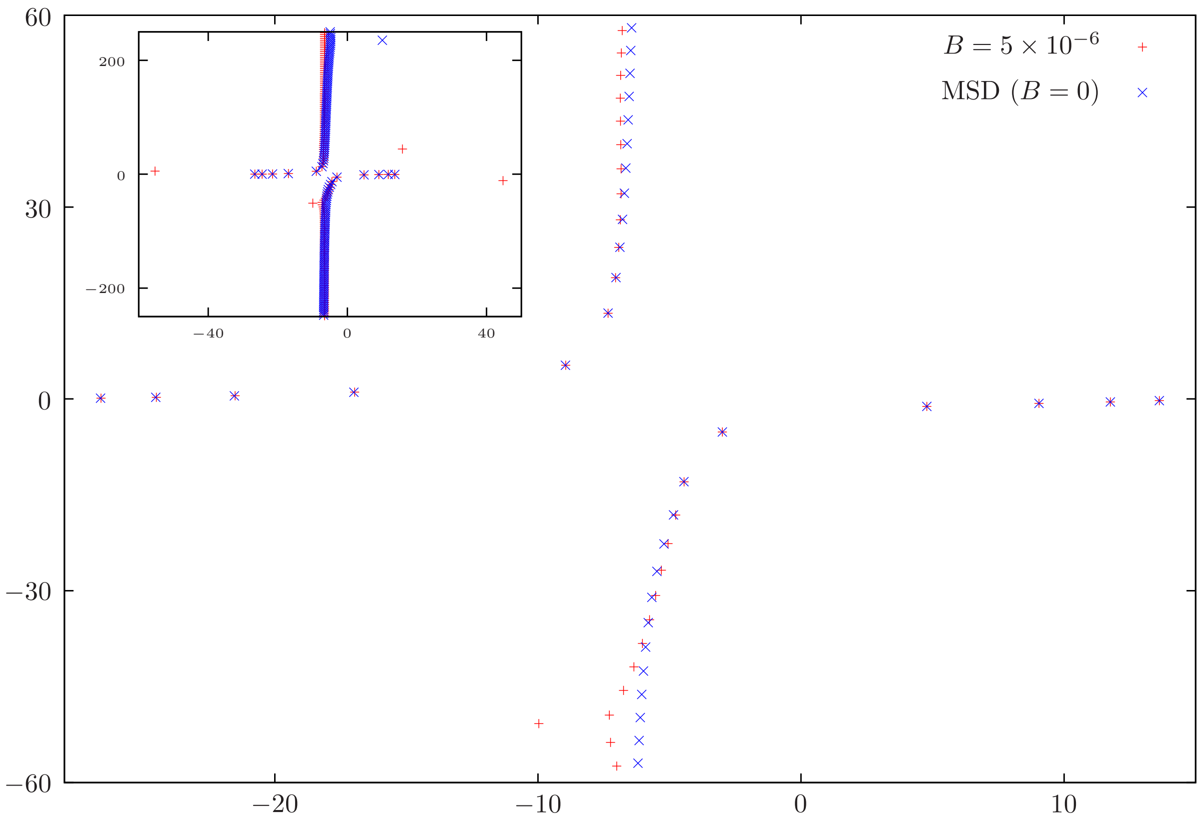

Figure 5. Modes in the k-plane for $U = 0.3$, $m = 5$, $\omega = 20$, $Z = 1 + 0.6\mathrm{i} - \mathrm{i}Bk^4/\omega$. The inset shows the locations of the surface modes.

V. An empirical wellposed impedance boundary model

We now propose the non-locally-reacting impedance boundary model

$$Z(k,\omega) = Z_{\mathrm{LR}}(\omega) - \mathrm{i}Bk^4/\omega, \tag{10}$$

where Z_{LR} is a locally-reacting impedance boundary model of the lining, and B is a *bending stiffness* coefficient, which is typically very small (of the order of 10^{-6}). The aim of this bending stiffness is to regularize the problem and make it wellposed without altering its behaviour for small or moderate k. This is motivated by using an MSD locally-reacting impedance boundary, giving a simplified version of the thin shell boundary analysed in detail in Ref. 16,

$$d\frac{\partial^2 w}{\partial t^2} + R\frac{\partial w}{\partial t} + bw + B\frac{\partial^4 w}{\partial x^4} = p, \qquad \Rightarrow \qquad Z(k,\omega) = R + \mathrm{i}d\omega - \mathrm{i}b/\omega - \mathrm{i}Bk^4/\omega, \tag{11}$$

where w is the normal displacement of the boundary (the vortex-sheet). The thin shell boundary was shown to be wellposed and causal for any $B > 0$, and stable provided B was sufficiently large (Ref. 16 p. 412). Reworking that stability analysis for (11) suggests a stable solution provided

$$B > \frac{3^3\, U^8}{4^4\,(1-U^2)^2\, b^3}. \tag{12}$$

As an example of the effect of the bending stiffness term, figure 5 shows the values of k for modes of the form $\exp\{\mathrm{i}\omega t - \mathrm{i}m\theta - \mathrm{i}kx\}$ for $\omega = 20$ and $m = 5$, for the impedance given in (11) with $R = 1$, $b = 8$, $d = 0.05$ and $B = 5 \times 10^{-6}$, and the equivalent MSD impedance $Z = 1 + 0.6\mathrm{i}$. Since $B > 1.6 \times 10^{-8}$, the value given by (12), this value of B renders the problem stable. The inset to figure 5 shows that the bending stiffness introduces four new surface modes, and also that it removes the MSD hydrodynamic instability surface mode.

Other than these surface modes, most modes are identically located for either boundary owing to the small value of B used. In fact, the bending stiffness supports up to ten surface modes, rather than just the four predicted for a locally-reacting boundary,[8,18] since the asymptotic surface-mode dispersion relation (9) in this case becomes a 10th order polynomial in k (see Ref. 16 for a detailed description of these surface modes). These surface modes are part of the entire problem, and so should be included in any scattering or mode-matching problems, where their effect may be significant despite their stable exponentially-decaying nature.

Larger values of B cause the effect of the bending stiffness to be felt at longer wavelengths, which may be beneficial for numerical simulation by allowing a coarser grid, although larger values of B will also cause the acoustic modes to be more displaced from their locally-reacting-boundary counterparts.

VI. Conclusion

The so-called "Crighton–Leppington" criterion has no rigorous mathematical derivation, and has been shown to incorrectly predict the stability of the wave-like differential equation example 1. Its use should therefore be strongly discouraged. The term "Crighton–Leppington" criterion has only appeared recently,[4,20] although the criterion has been in use before this.[21] It should be emphasized that this criterion is not due to Crighton or Leppington, nor to Jones or Morgan (to whom it was attributed in Ref. 21). As was mentioned in the introduction, both Jones & Morgan[12] and Crighton & Leppington[11] never relied on their time-harmonic analysis, but instead gave explicit causal solutions in terms of ultradistributions.

The problem of acoustics in lined ducts with mean flow has been shown to be illposed for several lining models (the MSD, HR and EHR models). Being illposed implies that there is no regular mathematical solution to the problem, essentially because the equation permits exponential growth $\exp\{\eta t\}$ for arbitrarily large η, and so the solution at time t in the limit $t \to 0$ does not necessarily coincide with the initial conditions. Not having a regular mathematical solution makes it impossible to analyse stability. Since there is no regular mathematical solution to expand in terms of duct modes or single frequencies, such analyses are at best contradictory, and at worst invalid, causing problems with frequency-domain simulations, mode-matching, and scattering. Being illposed with $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$ also means that numerical simulations in the time domain become unstable at the grid scale whatever grid scale is used, since the problem possesses arbitrarily-quick growth at arbitrarily-short wavelengths. Simply filtering out the instability is not satisfactory, since there is no reason the filtered numerical solution should be connected to the underlying physical system being modelled.

Whilst we have shown several lining models to be illposed, this was because all had the behaviour $\mathrm{i}\omega Z \sim A\omega^\sigma$ for $A < 0$ and $\sigma > 1$, where in all cases considered, $A = -d$ was the mass reactance and $\sigma = 2$. Setting $d = 0$ in these models leads to a different regime, which may or may not be wellposed. However, such models, especially the MSD model, do not have the flexibility to give an arbitrary target impedance Z, while still adhering to Rienstra's rules on admissible impedances.[2] Note also that to ensure a stable solution, the condition for B given in (12) involves the spring reactance b, rather than the mass reactance d, showing that both must be important to stability.

The illposed problem of perturbations to a vortex sheet was regularized by Jones[17] by considering a shear layer of finite thickness h, with the previous behaviour[11,12] recovered in the limit $h \to 0$. This is comparable with the regularization of including a bending stiffness proposed here in (10), where we recover the locally-reacting impedance model in the limit $B \to 0$, although the regularization here is empirical and not motivated by any physical phenomenon. We have suggested that the MSD boundary regularized in this way is wellposed for $B > 0$, and stable provided B is sufficiently large. It might be expected, although has not been shown here, that such a regularization also yields a wellposed stable problem provided B is sufficiently large when other locally-reacting impedances are regularized in this way, although how large B should be in such cases is undetermined here.

For practical use, we would recommend using the regularization (11) in numerical work, both in the time domain, and in the frequency domain for scattering and mode-matching problems, in which all modes including surface modes should be included and considered stable provided B is sufficiently large.

For theoretical work on locally-reacting impedances, we would also recommend an additional condition be added to Rienstra's four conditions for an impedance Z to be admissible,[2] namely that there should be a lower bound η such that any solution of (9) for $k \in \mathbb{R}$ should necessarily have $\mathrm{Im}(\omega) > \eta$; this constraint enforces wellposedness. Note that these conditions, as expressed in Ref. 2, are not applicable if the impedance is not locally-reacting.

Of course, the model (11), nor the underlying regularization (10), are attempting to model the actual physics of an acoustic lining; neither was the MSD model when it was proposed. Rather, they give a model with parameters that can be fitted to known experimental impedance data to give an empirically-correct, yet relatively simple, impedance model for practical use. A wellposed model of the actual physics would obviously be preferable. It has been suggested[1, 13, 22] that modelling the shear layer over the lining is important, rather than the assumption of a vortex sheet, and this would seem to correlate well with Jones'[17] regularization of the Kelvin–Helmholtz instability. However, such a model brings with it its own problems, such as the hydrodynamic continuous spectrum (not modelled in Ref. 1) and thereby the lack of a solution as a sum of duct modes.

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