Low-frequency acoustic reflection at a hard-soft lining transition in a cylindrical duct with uniform flow

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Abstract.
This paper considers the low-frequency limit of the reflection coefficient for downstream-propagating sound in a cylindrical duct with uniform mean flow at a sudden hard–soft wall impedance transition. The scattering at such a transition for arbitrary frequency was analysed by Rienstra (2007, J. Engng Maths 59, pp. 451–475), who, having derived an exact analytic solution, also considered the plane-wave reflection coefficient, R_{011}, in the low-frequency limit, and it is this result that is reconsidered here. This reflection coefficient was shown to be significantly different with or without the application of a Kutta-like condition and the corresponding inclusion or exclusion of an instability wave over the impedance wall, assuming an impedance independent of frequency. This analysis is here rederived for a frequency-dependent locally-reacting impedance, and a dramatic difference is seen. In particular, the Kutta condition is shown to have no effect on R_{011} in the low-frequency limit for impedances with $Z(\omega) \sim -\mathrm{i}b/\omega$ for some $b > 0$ as $\omega \to 0$, which includes the mass–spring–damper and Helmholtz resonator impedances, although, interestingly, not the enhanced Helmholtz resonator model. This casts doubt on the usefulness of the low-frequency plane-wave reflection coefficient as an experimental test for the presence of instability waves over the surface of impedance linings.

The plane-wave reflection coefficient is also derived in the low-frequency limit for a thin shell boundary, based on the scattering analysis of Brambley & Peake (2008, J. Fluid Mech. 602, pp. 403–426), who suggested the model as a well-posed regularization of the mass–spring–damper impedance. The result might be interpretable as evidence for the nonexistence of an instability over an acoustic lining.

Keywords: Impedance boundary, acoustic lining, Myers' boundary condition, scattering, low-frequency asymptotics.

1. Introduction

Rienstra [1] considered a cylindrical duct containing mean flow in the positive x-direction, which for $x < 0$ had a rigid boundary and for $x > 0$ had an acoustically lined boundary. The scattering of a downstream-propagating mode was considered as it encountered the sudden impedance change at $x = 0$, and a full analytic solution was derived using the Wiener–Hopf technique. Due to the debate over whether one of the surface modes constituted a downstream-propagating instability or an upstream-propagating evanescent mode, both were allowed for using a factor Γ: setting $\Gamma = 1$ gave the "full Kutta condition" of smooth $O(x^{3/2})$ behaviour of the surface streamline at $x = 0$ by including the surface mode as an instability, whereas setting $\Gamma = 0$ assumed the surface mode to be stable and upstream-propagating and gave an $O(x^{1/2})$ cusp in the surface streamline at $x = 0$.

The situation we consider is exactly that analysed by Rienstra [1], and is shown schematically in figure 1. We consider a straight cylindrical duct with centreline along the x-axis and constant radius ℓ^* (* is used throughout to denote a dimensional variable), the cross-section being described by polar coordinates (r^*, θ). Along this duct flows a uniform inviscid compressible mean flow in the positive x-direction of velocity U^*, with constant sound speed c^*, constant density ρ_0^*, and constant pressure p_0^*. On top of this mean flow we consider acoustic waves, so that the total velocity is $\mathbf{U}^* = U^*\mathbf{e_x} + \nabla\phi^*$, the total pressure is $P^* = p_0^* + p^*$ and the total density is $D^* = \rho_0^* + \rho^*$, where p^*, ρ^* and ϕ^* are a small acoustic perturbation and have time dependence $\exp\{\mathrm{i}\omega^* t^*\}$.

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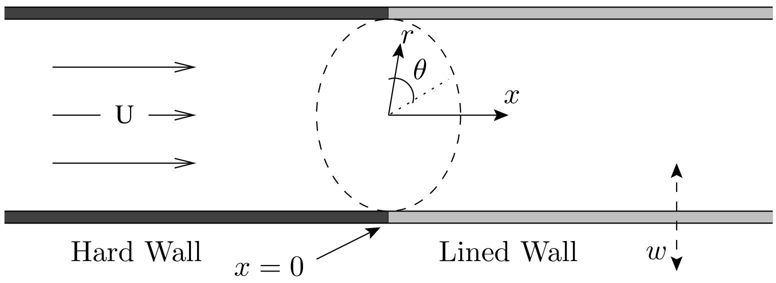

Figure 1. Sketch of the geometry

In order to reduce the number of constants, we now nondimensionalize based on c^*, ℓ^*, and ρ_0^*. That is, we set

$$\begin{aligned} x^* &= x\ell^*, & r^* &= r\ell^*, & t^* &= t\ell^*/c^*, & U^* &= Mc^*, \\ \rho^* &= \rho\rho_0^*, & p^* &= p\rho_0^* c^{*2}, & \phi^* &= \phi c^*\ell^*, & \omega^* &= \omega c^*/\ell^*. \end{aligned}$$

Here, M is the Mach number and ω is the Helmholtz number. The nondimensionalized mean flow pressure then becomes $p_0 = 1/\gamma$, where γ is the ratio of specific heats.

Substituting $\mathbf{U}$, P, and D into the Euler equation and taking only terms linear in the small quantities yields the convected Helmholtz equation

$$\left(\mathrm{i}\omega + M\frac{\partial}{\partial x}\right)^2\phi - \nabla^2\phi = 0, \qquad\qquad p = \rho = -\left(\mathrm{i}\omega + M\frac{\partial}{\partial x}\right)\phi. \tag{1}$$

The boundary condition at the duct wall is continuity of particle displacement [2]. If the outward radial deflection of the surface streamline is $w^* = w\ell^*$ (as indicated in figure 1), this boundary condition is

$$\frac{\partial\phi}{\partial r} = \left(\mathrm{i}\omega + M\frac{\partial}{\partial x}\right)w \qquad \text{at } r = 1. \tag{2}$$

For $x < 0$, the duct wall is hard, and so $w \equiv 0$, implying $\partial\phi/\partial r = 0$ at $r = 1$. For $x > 0$ the boundary is modelled by an impedance $Z^*(\omega^*) = Z(\omega)\rho_0^* c^*$, where a harmonic pressure forcing p^* excites a harmonic wall velocity $\mathrm{i}\omega^* w^* = p^*/Z^*$. Eliminating w from (2) gives the Myers boundary condition [3] (as applied to a flat boundary with uniform mean flow)

$$\mathrm{i}\omega\frac{\partial\phi}{\partial r} = \left(\mathrm{i}\omega + M\frac{\partial}{\partial x}\right)\frac{p}{Z}. \tag{3}$$

This is exactly the situation considered in [1].

Some common models for the dependence of Z on ω will be considered here. The first is the mass–spring–damper model, as used by Rienstra [1, 4], which treats the boundary as a locally-reacting simple harmonic oscillator with mass per unit area a^*, spring constant per unit area b^*, and damping constant per unit area R^*. Hence, when forced by the fluid pressure p^*, the boundary behaves according to

$$a^*\frac{\partial^2 w^*}{\partial t^{*2}} + R^*\frac{\partial w^*}{\partial t^*} + b^* w^* = p^*, \qquad \Rightarrow \qquad Z(\omega) = R + \mathrm{i}a\omega - \mathrm{i}b/\omega, \tag{4}$$

where

$$a^* = a\rho_0^*\ell^*, \qquad R^* = R\rho_0^* c^*, \qquad b^* = b\rho_0^* c^{*2}/\ell^*.$$

The justification of this model is that it captures three physical quantities: the inertia or mass a, the springiness b, and the damping R.

Another model is the enhanced Helmholtz resonator model [5], which is an extension of a model of a typical acoustic lining consisting of an array of Helmholtz resonators behind a perforated facing sheet. For this model

$$Z(\omega) = R + \mathrm{i}a\omega - \mathrm{i}Y\cot(\omega L - \mathrm{i}\varepsilon/2), \tag{5}$$

where L is the depth of the Helmholtz resonators[1], Y is a parameter scaling the cavity reactance, and ε is a damping within the fluid in the cavity. Setting $Y = 1$ and $\varepsilon = 0$ yields the original Helmholtz resonator model.

Separable (modal) solutions of (1) subject to (3) for a duct uniform for all x are of the form

$$\phi_{m\mu} = A_{m\mu}\mathrm{J}_m(\alpha_{m\mu}r)\exp\{\mathrm{i}\omega t - \mathrm{i}k_{m\mu}x - \mathrm{i}m\theta\}, \qquad \text{where } {\alpha_{m\mu}}^2 = (\omega - Mk_{m\mu})^2 - {k_{m\mu}}^2,$$

where J_m is the mth order Bessel function of the first kind. The boundary condition places the restriction on $k_{m\mu}$ that

$$\mathrm{i}\omega Z\alpha_{m\mu}{\mathrm{J}_m}'(\alpha_{m\mu}) = (\omega - Mk_{m\mu})^2\mathrm{J}_m(\alpha_{m\mu}).$$

There are an infinite discrete set of solutions (modes) $k_{m\mu}$ to this condition, which we index by the integer $\mu = 1, 2, \ldots$. Taking the limit $Z \to \infty$ regains the hard-wall boundary condition ${\mathrm{J}_m}'(\alpha_{m\mu}) = 0$. For finite Z, most modes are acoustic modes and have nearly real $\alpha_{m\mu}$. Rienstra [4] identified and characterized *surface modes* localized close to the duct boundary, for which $\alpha_{m\mu}$ has a large imaginary part (this analysis having been subsequently extended in [6]). Using the mass–spring–damper lining model, Rienstra tentatively suggested that one of these surface modes, present only with nonzero mean flow, might be interpreted as a downstream-growing instability, and termed this mode a hydrodynamic instability (HI) mode. It is this mode whose inclusion or exclusion was shown in [1] to have a significant effect on the plane-wave reflection coefficient R_{011} in the low-frequency regime.

In this paper (as seems to have been considered in [1]) we consider the low-frequency limit to be $\omega^* \to 0$ with ℓ^* and c^* fixed. This is subtly different from a general small-Helmholtz-number limit[2]. The distinction is that the Helmholtz number may be made small by reducing ℓ^* while holding ω^* and c^* fixed. For a mass–spring–damper boundary, if the boundary were unaltered during this radius change then a^*, b^* and R^* would remain fixed, and so the nondimensionalized variables a, b and R would vary with the Helmholtz number (the same is true for the nondimensionalized variables in the Enhanced Helmholtz Resonator model). Here we exclude this possibility. To give a physical perspective, what is imagined is an experiment in which we are given a lined duct that is to be investigated. This duct has a fixed radius (which is used to nondimensionalize). An acoustic wave is excited, say by loudspeakers, and the properties of the lining are then experimentally investigated by varying the frequency of the acoustic wave. This experiment is of a type that is commonly performed [7, 8, 9].

Introduced in [4] and used in [1], Rienstra defines the *reduced variables*

$$\beta^2 = 1 - M^2, \qquad \Omega = \omega/\beta, \qquad \sigma = M + \beta^2 k/\omega, \qquad \gamma^2 = 1 - \sigma^2 = \Big((\omega - Mk)^2 - k^2\Big)\beta^2/\omega^2,$$

where $\mathrm{Im}(\gamma) \le 0$ is taken. (Note that γ is not the ratio of specific heats defined earlier.)

2. The scattering problem and its solution

This section gives a brief review of the parts of the scattering analysis given in [1] that we will use subsequently. Consider a downstream-propagating mode incident from $x < 0$ of the form

$$p_{\mathrm{in}} = \mathrm{J}_m(\alpha_{m\mu}r)\exp\{\mathrm{i}\omega t - \mathrm{i}k_{m\mu}x - \mathrm{i}m\theta\}, \qquad \text{where } {\alpha_{m\mu}}^2 = (\omega - Mk_{m\mu})^2 - {k_{m\mu}}^2,$$

[1] Note that $\omega L = \omega^* L^*/c^*$, where $L^* = L\ell^*$, and so is independent of the duct radius, as expected.
[2] The author is grateful to a referee for pointing out this distinction.

where the hard-wall boundary condition places the condition on $k_{m\mu}$ that $\mathrm{J}_m{}'(\alpha_{m\mu}) = 0$ with solutions indexed by the integer μ. We now consider ω, m, and μ as given and fixed. Since p_{in} does not satisfy the boundary condition for $x > 0$, it scatters into waves propagating outward from $x = 0$. Rienstra [1] gave an exact analytic solution to this problem, using a Wiener–Hopf technique. For $x < 0$, the solution is given as a sum of hard-wall duct modes [1, equ. 34],

$$p = p_{\mathrm{in}} + \sum_{\nu=1}^{\infty} R_{m\mu\nu}\mathrm{J}_m(\alpha_{m\nu}r)\exp\{\mathrm{i}\omega t - \mathrm{i}k_{m\nu}x - \mathrm{i}m\theta\},$$

where $k_{m\nu}$ are the upstream-propagating modes in the hard-wall duct which therefore satisfy $\mathrm{J}_m{}'(\alpha_{m\nu}) = 0$. The reflection coefficient of a downstream-propagating plane wave $(m,\mu) = (0,1)$ into an upstream-propagating plane wave is given by [1, equ. 36]

$$R_{011} = \frac{1+M}{1-M}\frac{N_-(1)}{N_+(-1)}\left[\frac{1}{2} - \frac{\Gamma}{1+\sigma_{HI}}\right], \tag{6}$$

where σ_{HI} is the value of σ for the surface mode that is a potential candidate for an instability [4], and Γ is a Kutta-condition factor. $\Gamma = 0$ corresponds to σ_{HI} being considered stable and not present in $x > 0$, and leads to a boundary streamline cusp at $x = 0$ of $w = O(x^{1/2})$. $\Gamma = 1$ corresponds to a Kutta-like condition giving $w = O(x^{3/2})$, and necessitates the inclusion of the σ_{HI} mode as an instability in $x > 0$. $N_\pm(\sigma)$ are the split functions of the Wiener–Hopf kernel $K(\sigma) = N_+(\sigma)/N_-(\sigma)$, defined following [10] as

$$\log N_\pm(\sigma) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log K(u)}{u-\sigma}\,\mathrm{d}u, \tag{7}$$

where the integration contour is deformed above σ for N_- and below σ for N_+. The Wiener–Hopf kernel $K(\sigma)$ is given by [1, equ. 21]

$$K(\sigma) = \frac{\omega^2}{\beta^4}(1-M\sigma)^2\frac{\mathrm{J}_m(\alpha)}{\alpha\mathrm{J}_m{}'(\alpha)} - \mathrm{i}\omega Z, \tag{8}$$

where $\alpha^2 = (\omega - Mk)^2 - k^2 = (1-\sigma^2)\omega^2/\beta^2$.

In §6 of [1], Rienstra considered the low-frequency asymptotics of R_{011}. In doing so, it was assumed [1, equ. 59] that

$$K(\sigma) = -2\frac{(1-M\sigma)^2}{\beta^2\gamma(\sigma)^2} + O(\omega) \qquad \text{as } \omega \to 0, \tag{9}$$

which assumes that $Z(\omega) = O(1)$ as $\omega \to 0$. In this limit[3], $K(\sigma)$ can be factorized by inspection (taking care to attribute the poles and zeros of $K(\sigma)$ to the correct halves of the σ-plane), to give [1, equ. 61]

$$K(\sigma) = \frac{N^+(\sigma)}{N^-(\sigma)}, \qquad N^+(\sigma) = -2\frac{1-M\sigma}{\beta^2(1-\sigma)}, \qquad N^-(\sigma) = \frac{1+\sigma}{1-M\sigma}.$$

Rienstra identified one of the modes of the double-root at $\sigma = M^{-1}$ as the σ_{HI} mode. Substituting $\sigma_{HI} = M^{-1}$ and N^+ and N^- from above into (6) gives [1, equ. 62]

$$R_{011} \to -\frac{1+M}{1-M}\left(1 - \frac{2M\Gamma}{1+M}\right) \qquad \text{as } \omega \to 0.$$

Hence, $R_{011} \to -1$ for the full Kutta condition of $\Gamma = 1$ (including an instability), whereas $R_{011} \to -(1+M)/(1-M)$ for the stable solution $\Gamma = 0$.

[3] The validity of deriving approximations to $N_\pm(\sigma)$ using approximations for $K(\sigma)$ was proven by Koiter [11].

However, in §5 of [1] concerning causality, Rienstra used the mass–spring–damper impedance $Z = R + \mathrm{i}a\omega - \mathrm{i}b/\omega$, so that $Z = O(1/\omega)$ as $\omega \to 0$. For this type of impedance, (9) is no longer valid. In order to investigate what happens in this case, we now rederive the behaviour of R_{011} as $\omega \to 0$ for such an impedance.

3. Low-frequency asymptotics for a frequency-dependant impedance

Using either the mass–spring–damper model (4) or the Helmholtz resonator model (5, with $\varepsilon = 0$, $Y = 1$, $L = 1/b$) gives $Z(\omega) = -\mathrm{i}b/\omega + O(1)$ as $\omega \to 0$. Physically, this is saying that at low frequency the boundary's response is dominated by a spring-like force. For either of these models, (9) should be replaced with

$$K(\sigma) = -2\frac{(1 - M\sigma)^2}{\beta^2\gamma(\sigma)^2} - b + O(\omega). \tag{10}$$

Factorizing this by inspection, and taking care to attribute the poles and zeros of $K(\sigma)$ to the correct halves of the σ-plane, leads to the split kernels

$$K(\sigma) = \frac{N^+(\sigma)}{N^-(\sigma)}, \qquad N^+(\sigma) = -2\left(1 - \frac{\beta^2 b}{2M^2}\right)\frac{M\sigma^+ - M\sigma}{\beta^2(1-\sigma)}, \qquad N^-(\sigma) = \frac{1+\sigma}{M\sigma^- - M\sigma}.$$

where

$$\sigma^\pm = \frac{2M \mp \beta^2\sqrt{(2+b)b}}{2M^2 - \beta^2 b}.$$

Note that this reduces to the split kernels in the previous section on setting $b = 0$. Substituting these into (6) gives

$$R_{011} \to -\frac{1+M}{1-M}\Big(1 + b + \sqrt{(2+b)b}\Big)^{-1}\Big(1 - \frac{2\Gamma}{1+\sigma_{HI}}\Big) \qquad \text{as } \omega \to 0. \tag{11}$$

Following the same argument as used above to deduce σ_{HI} would give $\sigma_{HI} \to \sigma^-$ as $\omega \to 0$. However, it turns out that this is not true if $b \neq 0$, at least in the case we consider below. Using the asymptotics of Bessel functions [12, p. 364] that $\mathrm{J}_0(\alpha)/\mathrm{J}_0'(\alpha) \to \mathrm{i}$ as $|\alpha| \to \infty$ with $\mathrm{Im}(\alpha) > 0$, the full Wiener–Hopf kernel (8) may be expanded to leading order to gives a different balance of terms when $b \neq 0$, yielding a root when

$$\sigma \sim \frac{\beta^3 b}{M^2\omega} \qquad \text{as } \omega \to 0.$$

This we identify as σ_{HI}, which is verified numerically for the case given below. Since $\sigma_{HI} = O(1/\omega)$, the term involving Γ in (11) tends to zero as $\omega \to 0$, so that, to leading order, there is no effect of Γ on R_{011}. We conclude that the value of R_{011} in the limit $\omega \to 0$ is not sufficient to differentiate between stable and unstable behaviours of the HI surface mode for impedances of mass–spring–damper or Helmholtz-resonator type.

As an example to illustrate this, figure 2 compares the impedances $Z = 1 - 2\mathrm{i}$ (reproducing exactly figures 9 and 10 of [1]) and a typical low-frequency impedance $Z = 1 - 2\mathrm{i}/\omega$, calculated by numerically integrating the exact analytic solution derived in [1] and given here by (6), (7), and (8). Plotted are $|R_{011}|$ and the end correction δ_{011}, defined as [1, equ. 64]

$$\delta_{011} = (1 - M^2)\frac{\pi - \arg(R_{011})}{2\omega}$$

The values predicted by (11) are $R_{011} \approx -0.5147$ irrespective of Γ. This agrees well with the numerics plotted in figure 2, which for $\omega = 0.025$ gives $R_{011} \approx -0.5204 + 0.0017\mathrm{i}$ for $\Gamma = 0$, and $R_{011} \approx -0.5150 + 0.0016\mathrm{i}$ for $\Gamma = 1$.

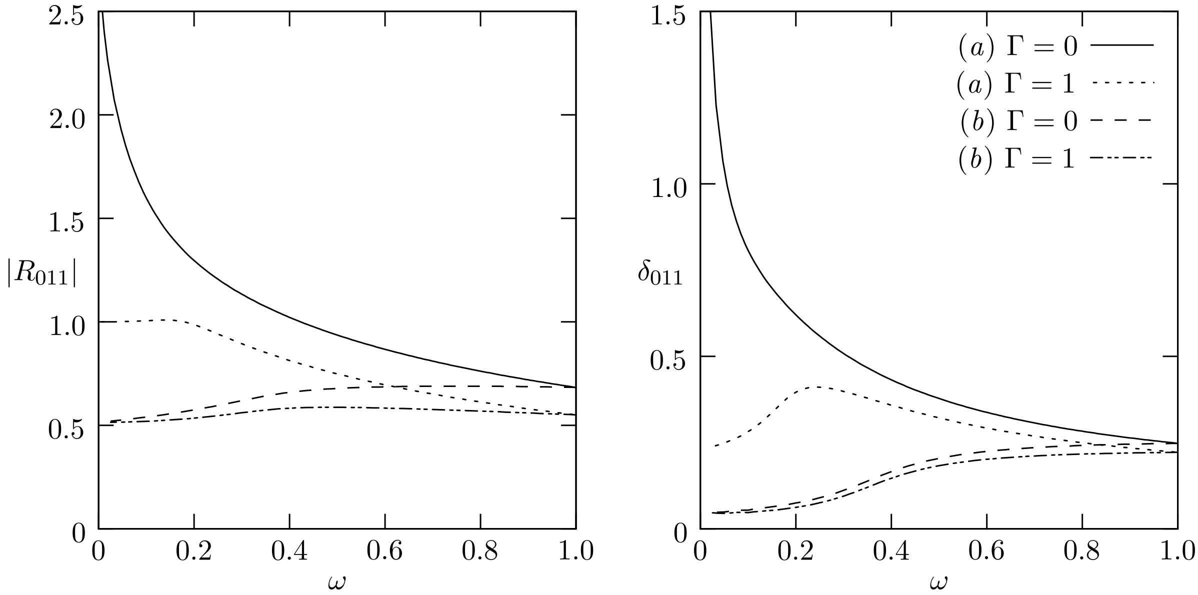

Figure 2. Plane-wave reflection coefficients for small ω for $M = 0.5$ and two impedance models: (a) $Z = 1 - 2\mathrm{i}$ (reproducing figures 9 and 10 of [1]); (b) $Z = 1 - 2\mathrm{i}/\omega$.

In a previous paper [13], the author argued that the mass–spring–damper impedance is illposed, leading to several apparent paradoxes including the stability nature of the σ_{HI} surface mode. A suggested resolution of this problem was to use the wellposed model of a cylindrical thin shell, which recovers the mass–spring–damper behaviour in the limit of the shell thickness $h \to 0$. We now investigate the low-frequency scattering for this boundary.

4. Low-frequency asymptotics for a thin-shell impedance

In [13], a cylindrical duct was considered which for $x < 0$ had a rigid boundary and for $x > 0$ had a boundary modelled as a thin shell clamped at $x = 0$. An exact analytic solution was given to the scattering of a wave in the hard section encountering the sudden boundary change at $x = 0$ using a Wiener–Hopf technique, in a similar manner to [1]. Here we are interested in plane-wave modes for which the azimuthal order is $m = 0$. Substituting this into equations 4.4 and 4.8 of [13], the Wiener–Hopf kernel we are interested in factorizing in this case (c.f. equation 8) becomes

$$K(k;\omega) = \frac{(\omega - Mk)^2 \mathrm{J}_0(\alpha)}{\alpha \mathrm{J}_0'(\alpha)} - \left(\tilde{b} + \mathrm{i}\omega R - d\omega^2 + Bk^4 + \mathrm{c}_l{}^2 dk^2 \frac{\left(\nu + h^2k^2/12\right)^2}{\omega^2/\mathrm{c}_l{}^2 - k^2} \right),$$

where $\tilde{b} = b + \mathrm{c}_l{}^2 d$ is the spring force, R is the damping, $B = \mathrm{c}_l{}^2 dh^2/12$ is the bending stiffness, c_l is the speed of sound in the thin shell, h is the thin shell thickness, $d = \rho_s h$ is the thin shell density per unit area, and ν is the Poisson ratio of the thin shell. All of these constants are considered to be nondimensional, as in previous sections. By analogy with the mass–spring–damper impedance, $(d, \tilde{b}, R)$ here play the role of (a, b, R) for the mass–spring–damper impedance, while the bending stiffness B regularizes the problem and the final term describes the interaction between the fluid and waves in the thin-shell boundary. The mass–spring–damper impedance is recovered by taking $\nu = 0$ and the limit as $h \to 0$ with $d = O(1)$.

The reflection coefficients are given by equation 4.13 of [13]. We are interested in the plane-wave reflection of a plane incident wave, and so we set $m = 0$, $k_{jm} = -\omega/(1 - M)$ and

$k_{\rm in} = \omega/(1+M)$, to get

$$R_{011} = \frac{(1+M)K^-\Big(\omega/(1+M)\Big)}{2(1-M)K^+\Big(-\omega/(1-M)\Big)}\left[1+\frac{{\rm c}_l{}^3 d\Big(\nu+\frac{\omega^2h^2}{12{\rm c}_l{}^2}\Big)^2}{\Big({\rm c}_l{}^2+2{\rm c}_l+\beta^2\Big)\Big(\frac{K^+(-\omega/{\rm c}_l)}{K^-(\omega/{\rm c}_l)}-\frac{1}{4}{\rm c}_l{}^2 d(\nu+\frac{\omega^2h^2}{12{\rm c}_l{}^2})^2\Big)}\right]. \tag{12}$$

As it stands, this equation is exact, but difficult to interpret. We therefore look at the limit $\omega \to 0$, in order to derive the comparable result to (6) and (11). To do this, we must evaluate the factorized kernels K^+ and K^- in the low-frequency limit. As in the previous section, we do this by looking at the asymptotics of $K(k;\omega)$ as $\omega \to 0$ and then factorize. Here, there are two regimes we must consider in order to give a uniformly-valid asymptotic approximation[4] for all k,

$$K(q\omega;\omega) = K_1(q) + O(\omega), \qquad K(k;\omega) = K_2(k) + O(\omega),$$

$$K_1(q) = \frac{-2(1-Mq)^2}{1-2Mq-\beta^2q^2} - \left(\tilde{b} - \frac{{\rm c}_l{}^2 d\,\nu^2}{1-\frac{1}{{\rm c}_l{}^2q^2}}\right),$$

$$K_2(k) = \frac{-{\rm i}M^2k{\rm J}_0({\rm i}\beta k)}{\beta{\rm J}_0'({\rm i}\beta k)} - \Big(\tilde{b} - {\rm c}_l{}^2 d\nu^2 - 2\nu Bk^2 + B(1-h^2/12)k^4\Big).$$

Naively, one might have expected to be able to use only K_1 as the asymptotic kernel, as in the previous sections. However, note that the bending stiffness term involving B only occurs in K_2. It was the bending stiffness term that was shown in [13] to be important for the wellposedness of the thin shell scattering problem, and so we will be reluctant to discard it here. Instead, we follow Crighton [14] and use a multiplicative composition, giving the uniform composite asymptotic expansion

$$K(k;\omega) = \frac{K_1(k/\omega)K_2(k)}{2M^2/\beta^2 - \tilde{b} + {\rm c}_l{}^2 d\nu^2} + O(\sqrt{\omega}) \qquad \text{as } \omega \to 0, \tag{13}$$

where we assume the constant denominator is nonzero. We may therefore split $K = K^+/K^-$, with $K^\pm$ being analytic and nonzero in the upper- and lower-half k-planes respectively, using the multiplicative splits of K_1 and K_2.

In calculating R_{011} in the limit $\omega \to 0$, we would like to assume that, for arbitrary q_1 and q_2,

$$\lim_{\omega\to 0}\frac{K^+(q_1\omega)}{K^-(q_2\omega)} = \frac{K_1^+(q_1)}{K_1^-(q_2)}. \tag{14}$$

This is not obvious a priori, as we would like to take the limits of K_2^+ and K_2^- independently and at different rates q_1 and q_2. Is it true, nevertheless, and this is shown in appendix A. Making use of this in (12) leads to the plane-wave reflection coefficient given, as might naively have been expected without this technicality, by

$$R_{011} = \frac{(1+M)K_1^-(1/(M+1))}{2(1-M)K_1^+(1/(M-1))}\left[1+\frac{{\rm c}_l{}^3 d\nu^2}{\Big({\rm c}_l{}^2+2{\rm c}_l+\beta^2\Big)\Big(\frac{K_1^+(-1/{\rm c}_l)}{K_1^-(1/{\rm c}_l)}-{\rm c}_l{}^2 d\nu^2/4\Big)}\right]. \tag{15}$$

In order to calculate $K_1^+(q)$ and $K_1^-(q)$, we locate the poles and zeros of $K_1(q)$ and then factorize by inspection. In general, the poles of $K_1(q)$ are at $q = 1/(M\pm 1)$ and $q = \pm 1/{\rm c}_l$, while the zeros of $K_1(q)$ are the solutions of a quartic equation. In general, therefore, (15) differs from the locally-reacting reflection coefficient (11). However, if the Poisson ratio $\nu = 0$, then $K_1(q)$ becomes the locally-reacting kernel $K(\sigma)$ given in (10) with $\sigma = M + \beta^2 q$, and (15) becomes (11) with $\Gamma = 0$ and $\tilde{b}$ replacing b.

[4] A uniform approximation is also technically necessary for the case considered in §3, although it turns out there, as it will do here, that neglecting the $k = O(1)$ factor (equivalently the $\sigma = O(1/\omega)$ factor) is justified.

This shows that, using the (wellposed) thin-shell boundary model with $\nu = 0$, exactly the same low-frequency reflection coefficient is derived as for the mass–spring–damper model if no Kutta condition is applied ($\Gamma = 0$), even in cases where $\tilde{b} = b = 0$ for which Rienstra's original analysis [1] is valid and the Kutta and no Kutta condition cases give different results. The surface streamline in this case is $O(x^2)$ [13], giving smoother behaviour than either the Kutta or no Kutta condition locally-reacting cases. Despite the bending stiffness B being the cause of the wellposedness of the thin-shell mode, the low-frequency reflection coefficient is independent of B.

5. Conclusion

In this paper, we have shown that the low-frequency analysis of the plane-wave reflection coefficient R_{011} by Rienstra [1] is only valid provided the impedance satisfies $\omega Z(\omega) \to 0$ as $\omega \to 0$. In the case of a mass–spring–damper or Helmholtz resonator impedance, $Z \sim -\mathrm{i}b/\omega$ for some $b > 0$, and so this is not the case. We have shown, in §3, that for such models the Kutta condition factor Γ disappears from the formula for R_{011} in the low-frequency limit, due to qualitatively different behaviour of the hydrodynamic "instability" mode σ_{HI}. We therefore conclude that the suggestion in [1] of experimentally investigating R_{011} in the low-frequency limit is not sufficient to determine the nature of the potential instability, at least for the impedances considered here.

In [1], Rienstra comprehensively analysed the scattering of sound in a duct encountering a sudden transition from hard-walled to acoustically lined, including the derivation of an exact analytical result. One section of this analysis [1, §6] was an investigation of the plane-wave reflection coefficient R_{011} in the low-frequency limit. It is emphasized that this paper is only concerned with that section of [1], and does not take issue with the significant remainder of that paper.

The limit considered here is motivated by a notional physical experiment in which the acoustic frequency ω^* is reduced to zero, with the other parameters remaining fixed. Due to the nondimensionalization of the lining parameters used here, the physics would be considerably different if the small Helmholtz number limit were reached by varying the duct radius ℓ^*, and may well give results closer to those originally predicted by [1].

It is notable that, for the enhanced Helmholtz resonator with $\varepsilon \neq 0$ given in (5), $Z(\omega) \to R + 2Y/\varepsilon$ as $\omega \to 0$, and so this impedance is bounded and the original analysis of [1] is valid; this behaviour being seen for $\omega \ll \varepsilon/2L$. The dramatic effect of a small but nonzero ε in the zero-frequency limit is surprising, and we do not pretend to explain here physically why this should be so. Indeed, the validity of any of the impedance models considered here (including the fixed-impedance model) in the zero-frequency limit remains, to the author's knowledge, unverified.

From the results of this analysis, it is interesting that the reflection coefficient R_{011} is independent of the lining resistance R, at least to leading order. This appears to indicate that the only relevant lining physics involved in plane wave reflection in the low-frequency limit is the spring-like behaviour of the lining. This could be expected to change if the liner were only of finite, rather than infinite, extent, since the resistance would then affect the standing waves within the lined section, some of which would bleed through the upstream impedance transition and contribute to the reflected wave.

If the mass–spring–damper model were to be considered as a thin shell of zero Poisson ratio in the limit of small bending stiffness (a singular limit of a wellposed boundary model), then we have seen in §4 that the no Kutta solution ($\Gamma = 0$) is the correct one, even for cases when $\tilde{b} = b = 0$ for which the analysis of [1] is valid and varying Γ would have a large effect on R_{011}. This could be interpreted as evidence towards the hydrodynamic "instability" surface mode σ_{HI} actually being stable. However, this evidence is tentative at best, and it is the author's view that illposed problems such as these are paradoxical and can not be rigorously analysed.

The analysis of §4 was complicated by the need for a uniform asymptotic approximation to the Wiener–Hopf kernel. The solution was as one might naively have expected, but this was only because $K_2^\pm(k)$, which are not expressible in closed form, could be proven to be finite and nonzero at $k = 0$. Had they had either a pole or a zero at the origin, this conclusion would not have held. A pole or zero at the origin would not have been unexpected, since $K_2(k)$ is only a valid asymptotic representation of the true Wiener–Hopf kernel $K(k)$ for $k \gg \omega$, and we know from the inner expansion of $K(q\omega) = K_1(q) + O(\omega)$ that $K(k)$ does indeed have poles and zeros in the neighbourhood of the origin ($k = O(\omega)$).

Appendix

A. Asymptotics of the Kernel Factorization

We are given the Wiener–Hopf kernel $K(k;\omega)$, with asymptotics

$$K(q\omega;\omega) = K_1(q) + O(\omega), \qquad K(k;\omega) = K_2(k) + O(\omega),$$

$$K_1(q) = \frac{-2(1-Mq)^2}{1-2Mq-\beta^2q^2} - \left(\tilde{b} - \frac{c_l^2 d\,\nu^2}{1-\frac{1}{c_l^2q^2}}\right),$$

$$K_2(k) = \frac{-\mathrm{i}M^2k\mathrm{J}_0(\mathrm{i}\beta k)}{\beta\mathrm{J}_0'(\mathrm{i}\beta k)} - \left(\tilde{b} - c_l^2 d\nu^2 - 2\nu Bk^2 + B(1-h^2/12)k^4\right),$$

and, following Crighton [14], we form the uniform composite asymptotic expansion

$$K(k;\omega) = \frac{K_1(k/\omega)K_2(k)}{2M^2/\beta^2 - \tilde{b} + c_l^2 d\nu^2} + O(\sqrt{\omega}) \qquad \text{as } \omega \to 0,$$

where

$$\lim_{q\to\infty} K_1(q) = \lim_{k\to 0} K_2(k) = 2M^2/\beta^2 - \tilde{b} + c_l^2 d\nu^2,$$

which we assume to be nonzero.

Part of the Weiner–Hopf procedure involves forming the split functions $K^+(k;\omega)$ and $K^-(k;\omega)$, where K^+ is analytic and nonzero in the upper-half k-plane and along the real k axis, and K^- is analytic and nonzero in the lower-half k-plane and along the real k axis. In general, K^+ and K^- must be calculated numerically, as in [13]. Here, however, we are only interested in the low-frequency limit $\omega \to 0$, and the value of R_{011} given by (12) in this limit. We are therefore not interested in general in K^+ and K^-, but in

$$\lim_{\omega\to 0}\frac{K^+(q_1\omega;\omega)}{K^-(q_2\omega;\omega)} = \frac{K_1^+(q_1)}{K_1^-(q_2)}\frac{K_2^+(q_1\omega)}{K_2^-(q_2\omega)}\frac{1}{(2M^2/\beta^2 - \tilde{b} + c_l^2 d\nu^2)}.$$

We now show that this is equal to $K_1^+(q_1)/K_1^-(q_2)$, justifying (14), by showing that $K_2^+(0)$ and $K_2^-(0)$ are finite and nonzero. If they were not, then the limit above may have had a different numerical value than $K_1^+(q_1)/K_1^-(q_2)$, or may not have existed at all.

As $|k| \to \infty$, $K_2(k) \to -\infty$, whereas $K_2(0) = 2M^2/\beta^2 - \tilde{b} + c_l^2 d\nu^2$. Therefore, if $\tilde{b}$ is sufficiently small, $K_2(k)$ has a zero on the real k axis. Note also that $K_2(k) = K_2(-k)$. We set

$$L(k) = \log\left(\frac{-K_2(k)}{B(1-h^2/12)(k^2+X^2)^2}\right)$$

for some positive real constant X, so that $L(k) \to 0$ as $|k| \to \infty$ and $L(0)$ is finite. If $K_2(k)$ has zeros on the real axis, we take the branch cuts of $L(k)$ to be in the upper-half k-plane for zeros with $k < 0$ and in the lower-half k-plane for zeros with $k > 0$, taken symmetrically so that $L(k) = L(-k)$. Then we may form the additive decomposition of $L(k)$ (following [10, 13]),

$$L(k) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\delta}^{\infty-\mathrm{i}\delta}\frac{L(\xi)}{\xi-k}\,\mathrm{d}\xi - \frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\delta}^{\infty+\mathrm{i}\delta}\frac{L(\xi)}{\xi-k}\,\mathrm{d}\xi,$$

where $|\mathrm{Im}(k)| < \delta < X$ and the contours are deformed around the branch cuts symmetrically. We call the first integral $L^+(k)$ and the second $L^-(k)$, so that $L = L^+ - L^-$ forms the additive decomposition, and hence setting $K_2^{\pm} = \pm \mathrm{i} \exp\{L^{\pm}\}(B(1 - h^2/12))^{\pm 1/2}(k \pm \mathrm{i}X)^{\pm 2}$ gives the multiplicative decomposition $K_2 = K_2^+/K_2^-$. Since $L(k) = L(-k)$, and since the contours were chosen symmetrically, $L^+(0) = L^-(0) = L(0)/2$ and so are finite. Hence, $K_2^+(0)$ and $K_2^-(0)$ are finite and nonzero, and by definition $K_2^+(0)/K_2^-(0) = K(0)$. Therefore, we finally arrive at the identity (14).

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