Review of acoustic liner models with flow

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In many situations, fluid flows over a surface, and small amplitude oscillations in the fluid flow are coupled to small amplitude oscillations of the surface. This paper gives a brief review of recent advances in the understanding of modelling such situations. One major application of this is to model sound within aeroengines with acoustically lined walls. The emphasis of this paper is on mathematical modelling and frequency-domain computing, although implications for time-domain computing are also be briefly mentioned.

1 Introduction

Consider a uniform flow of Mach number M over a flat boundary. On top of this uniform flow we impose small amplitude perturbations. These perturbations couple with the boundary, so that a fluid pressure perturbation $\tilde{p}\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ yields a boundary normal velocity $\tilde{v}\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$. Since the perturbations are small, $\tilde{p}$ and $\tilde{v}$ are related linearly, so that the response of the boundary is totally characterised by its impedance $Z(\omega, k) = \tilde{p}/\tilde{v}$. The boundary need not represent a physically impermeable surface, but could be any boundary of the fluid such as a porous plate or a boundary with another fluid. This paper considers the boundary condition used to impose a particular impedance on the fluid at such a boundary.

For definiteness in what follows, consider flow in a cylinder in the x-direction. Nondimensionalizing by the centreline density, the centreline sound speed and the cylinder radius, the mean flow is given by $U(r)\boldsymbol{e_x}$ and the density by $\rho(r)$, while the mean pressure is constant. Considering perturbations to this, Fourier transforming $t \to \omega$ and $x \to k$ and expanding in a Fourier series $\theta \to m$ using the $\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}$ sign convention for the inverse transform yields the Pridmore-Brown [1] equation

$$\tilde{p}'' + \left(\frac{2kU'}{\omega - kU} + \frac{1}{r} - \frac{\rho'}{\rho}\right)\tilde{p}' + \left((\omega - kU)^2\rho - k^2 - \frac{m^2}{r^2}\right)\tilde{p} = 0,$$
$$\tilde{v} = \mathrm{i}\tilde{p}'/((\omega - Uk)\rho), \tag{1}$$

where a prime denotes $\mathrm{d}/\mathrm{d}r$. If the flow is uniform, so that $\rho(r) \equiv 1$ and $U(r) \equiv M$, then (1) reduces to Bessel's equation, the solution to which that is regular at $r = 0$ is $\tilde{p} = \mathrm{J}_m(\alpha r)$ with $\alpha^2 = (\omega - Mk)^2 - k^2$. The impedance boundary condition remains to be applied at $r = 1$.

Applying an impedance boundary condition without flow is as simple as specify that $\tilde{p}/\tilde{v} = Z$ on the boundary. However, if the mean flow is slipping at the boundary, it is well known that this boundary condition must be modified [2–7]. If the fluid velocity at the boundary is $\boldsymbol{U} + \boldsymbol{u}\exp\{\mathrm{i}\omega t\}$, then the boundary condition becomes

$$\mathrm{i}\omega\boldsymbol{u}\cdot\boldsymbol{n} = (\mathrm{i}\omega + \boldsymbol{U}\cdot\boldsymbol{\nabla} - (\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U})\cdot\boldsymbol{n})p/Z, \tag{2}$$

where $\boldsymbol{n}$ is the boundary normal pointing out of the fluid. This follows from matching fluid and boundary normal displacement, rather than normal velocity, and is often known as the Myers, or sometimes Ingard–Myers, boundary condition. The form given in (2) is indeed due to Myers [7], whose contribution was the final term $(\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U})\cdot\boldsymbol{n}$ representing curvature of the boundary, which for straight boundaries considered here is identically zero. Excluding this term, (2) was first postulated by Miles [2, equation (3.3)'] and shortly afterward by Ingard [4], although the incorrect boundary condition was still in use a decade later [e.g. 8]. Again without the final curvature term, (2) was shown to be the correct asymptotic limit of a vanishingly-thin boundary layer by Eversman & Beckemeyer [5] and Tester [6], although the boundary layer needed to be extremely thin to attain this limit in some cases [6, 9]; this is an important point that we return to in §3. Using the boundary condition (2) for uniform flow in a cylinder gives the dispersion relation relating ω and k as

$$1 - \frac{(\omega - Mk)^2}{\mathrm{i}\omega Z}\frac{\mathrm{J}_m(\alpha)}{\alpha\mathrm{J}_m'(\alpha)} = 0, \tag{3}$$

where, as before, $\alpha^2 = (\omega - Mk)^2 - k^2$, and the branch chosen for α does not matter.

When the impedance boundary condition (2) is used with time domain numerics involving slipping flow the numerics are found to be unstable at the grid scale for sufficiently fine grids. In order to converge, such numerics always include artificial damping to filter out the instability [e.g. 10–15]. For frequency domain numerics, the problem of unstable numerics is replaced by the problem of choosing the direction of propagation of modes [16]; i.e. of choosing the causal solution, which is related to the mathematical problem of analysing the stability of flow over linings [17–19], discussed further in §4. Some experiments [20–22] also show an instability being excited in certain, but not all, situations, although this instability does not agree with that predicted by (3) for uniform flow [22], as discussed in §5. Before this, however, some simple models for the impedance Z are discussed.

2 Some boundary impedance models

We first consider one of the most simple physically-motivated models for an impedance $Z(k, \omega)$. Consider a theoretical boundary with normal displacement w. Suppose that the boundary is *locally reacting*, so that each position on the boundary moves independently from each other position (an assumption relaxed later), the motion being as if a damped spring acted on it. Then the equation of motion would be

$$d\frac{\partial^2 w}{\partial t^2} + R\frac{\partial w}{\partial t} + bw = p,$$

where d is the mass/density of the boundary, R is the damping constant, b is the spring constant, and p is the fluid pressure acting on the boundary. Fourier transforming, using $\tilde{v} = \mathrm{i}\omega\tilde{w}$, and dividing by $\tilde{v}$ gives

$$Z(k, \omega) = R + \mathrm{i}d\omega - \mathrm{i}b/\omega. \tag{4}$$

This is know as a mass–spring–damper impedance, or equivalently, as the three-parameter model [10].

Suppose now that in addition to the damped spring forcing the boundary also feels a tension T and a simplified bending stiffness B (see [23] for details). Then the same procedure gives

$$Z(k, \omega) = R + \mathrm{i}d\omega - \mathrm{i}b/\omega - \mathrm{i}Tk^2/\omega - \mathrm{i}Bk^4/\omega. \tag{5}$$

Note that the boundary is no longer locally reacting, and hence the impedance depends on k as well as ω.

Acoustic linings in aeroengines typically consist of a perforated sheet over a honeycomb core of resonators with a rigid backing plate [e.g. 8]. This is often modelled using a Helmholtz Resonator impedance model that accounts for resonators of a depth L. Rienstra [24] gave a more adaptable model that he showed was just as easy to use numerically, which he termed the Extended Helmholtz Resonator (EHR) model, given by

$$Z = R + \mathrm{i}d\omega - \mathrm{i}\nu\cot(\omega L - \mathrm{i}\varepsilon/2),$$

where $\nu = 1$ and $\varepsilon = 0$ for the standard Helmholtz Resonator.

Other models, including empirically and semi-empirically fitted models, also exist [e.g. 25, 26]. However, it turns out many of these do not correspond to physically possible impedances, as discussed next.

2.1 Admissibility of impedances

Rienstra [24] pointed out that not all locally-reacting impedances $Z(\omega)$ represent physically-realisable boundaries, and gave four conditions on an impedance $Z(\omega)$ to be physically-realisable. Since $\tilde{p}(\omega) = Z(\omega)\tilde{v}(\omega)$, in the time domain

$$p(t) = \int_{-\infty}^{\infty} z\,(t-\tau)\,v(\tau)\,\mathrm{d}\tau,$$

where $z(t)$ is the inverse Fourier transform of $Z(\omega)$. Since $p(t)$ cannot depend on $v(\tau)$ for $\tau > t$, Rienstra argued, $z(t) \equiv 0$ for $t < 0$, implying that $Z(\omega)$ is analytic for $\mathrm{Im}(\omega) < 0$. Note that this is due to our choice of the $\exp\{\mathrm{i}\omega t\}$ convention. Similarly, $v(t)$ cannot depend on $p(\tau)$ for $\tau > t$, so $Z(\omega)$ must also be nonzero for $\mathrm{Im}(\omega) < 0$. Moreover, as $p(t)$ and $v(t)$ are both real, so is $z(t)$, implying that $\overline{Z(\omega)} = Z(-\overline{\omega})$, where a bar denotes the complex conjugate. Finally, for the boundary not to be a source of energy, $\mathrm{Re}(Z(\omega)) \geq 0$ for real ω. These four points are given as theorem 1 of [24].

Rienstra [24] went on to show that several empirical and semi-empirical impedance models do not adhere to these conditions, and so are not physically realisable. However, he also stated that these conditions do not preclude other unwanted or unphysical behaviour of the impedance model, such as, for example, a Kelvin–Helmholtz-type instability of the flow over the boundary. We now see that this is in fact inherent in the boundary condition applying the impedance model, rather than the impedance model itself.

3 A wellposed boundary condition

Since the impedance boundary condition (2) is the limit of an infinitely thin boundary layer, when applied with slipping mean flow it exhibits a comparable instability to the Kelvin–Helmholtz free shear layer instability. This can be seen by considering a cylindrical lined duct with a mass–spring–damper boundary (4) and uniform flow and solving the initial value problem; that is, given an axial wavenumber k and azimuthal wavenumber m, solving (3) for the allowable frequencies ω of the duct modes. Figure 1 shows a plot of the complex values of ω as k is varied with k real using the same parameters as [27, figure 3]. The majority of the trajectories have small but slightly positive $\mathrm{Im}(\omega)$ and correspond to damped almost-propagating acoustic modes. The two modes with $\mathrm{Im}(\omega) < 0$ are the Kelvin–Helmholtz equivalent modes and represent instabilities. In fact, these modes have $-\mathrm{Im}(\omega) = O(k^{1/2})$, as shown in [19], and thus arbitrarily short wavelengths are unstable with arbitrarily large exponential growth rates. This is the cause of the numerical time domain instabilities when using (2), as mentioned in §1.

This unboundedness leads to further problems [19]. It not only means that no correct Briggs–Bers stability analysis [28, 29] is possible, but also that the mathematical problem is illposed. The solution is to find a wellposed problem, which has been achieved simultaneously by Rienstra & Darau [30, 31] and Brambley [27, 32]. Here we follow the more general [27], while noting that both approaches are reasonably comparable for linear shear layers. [27] follows Eversman [5, 9, 33] by considering a thin nonslipping boundary layer of width δ in otherwise uniform flow over an acoustic lining, to which (2) is the leading order solution, and finding the $O(\delta)$ correction terms. In a cylindrical geometry, this gives the boundary condition

$$\mathrm{i}\omega Z\Big[\tilde{v} + \frac{k^2+m^2}{\mathrm{i}(\omega-Mk)}\delta I_1\tilde{p}\Big] = \mathrm{i}(\omega - Mk)\Big[\tilde{p} + \mathrm{i}(\omega-Mk)\delta I_0\tilde{v}\Big], \quad (6)$$

where the integrals δI_0 and δI_1 give the $O(\delta)$ corrections, so that setting them to zero recovers the leading order boundary condition (2). These integrals are given by

$$\delta I_0 = \int_0^1 1 - \frac{(\omega - U(r)k)^2\rho(r)}{(\omega - Mk)^2}\,\mathrm{d}r,$$
$$\delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{(\omega - U(r)k)^2\rho(r)}\,\mathrm{d}r,$$

and since the integrands are only nonzero within the boundary layer, they indeed give a contribution of $O(\delta)$. [27] goes on to interpret δI_0 in terms of the mass, momentum and kinetic energy thicknesses of the boundary layer, δ_{mass}, δ_{mom} and δ_{ke}, which for a mass–spring–damper impedance (4) leads to an effective modified impedance $\hat{Z}$, where

$$\mathrm{i}\omega\hat{Z} = -\omega^2(d - \delta_{\mathrm{mass}}) + \mathrm{i}\omega R + b + k^2M^2\delta_{\mathrm{ke}} - 2\omega k M\delta_{\mathrm{mom}}.$$

The penultimate term represents a tension along the boundary (c.f. (5)), while the last term may be interpreted as a convection term. The importance of this effective impedance Z_{mod} is that it is nonlocal even though the underlying mass–spring–damper impedance is locally reacting. All of this is due to the δI_0 term. The physical interpretation of the δI_1 term remains illusive, but it is known that it is this δI_1 term that is responsible for making the problem wellposed, and it may well have connections with the critical layer in sheared flow [34–36].

The duct modes calculated using the boundary condition (6) are plotted in figure 1, together with a numerical solution to the Pridmore–Brown equation (1) without any other approximation. The only modes to be significantly affected by the change of boundary condition are the unstable ones, which still represent unstable modes but now with bounded growth rate (in this case, $\mathrm{Im}(\omega) > -4$). The problem is therefore well posed, allowing a stability analysis to be performed.

4 Stability and surface modes

Most disturbances within the fluid over an impedance boundary are acoustic modes damped as energy is absorbed

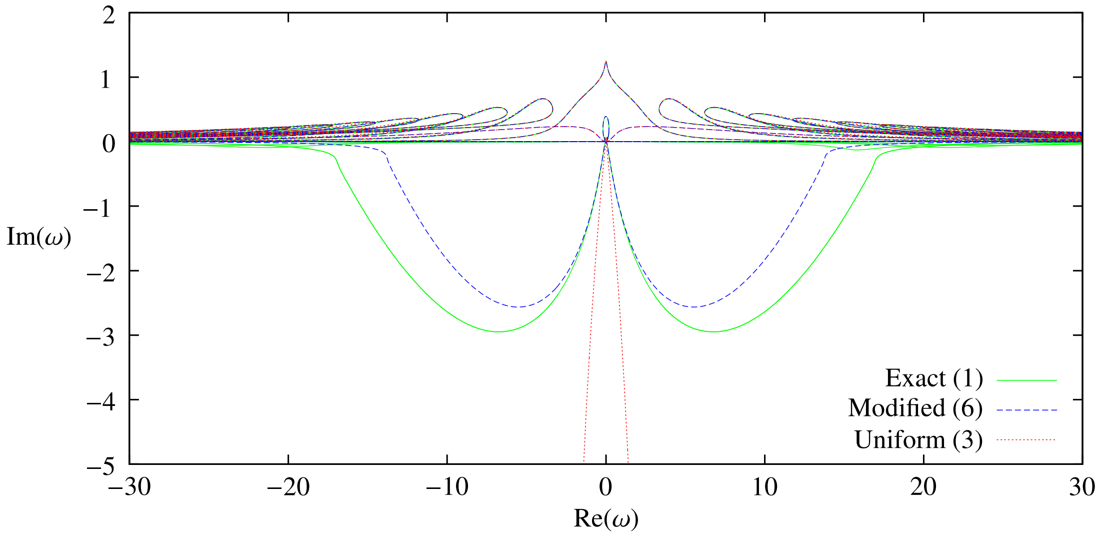

Figure 1: Trajectories in the ω-plane of $\omega(k)$ for $k \in \mathbb{R}$ for solutions to the Pridmore–Brown equation (1). Plotted are the exact numerically-calculated solutions and solutions using the modified boundary condition (6), both using $U(r) \approx M \tanh((1-r)/\delta)$, and the original boundary condition for uniform flow with $U(r) \equiv M$ given by (3). $M = 0.5$, $m = 0$, $\delta = 2 \times 10^{-3}$ and $Z = 3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$, the same parameters as in figure 3 of [27].

by the boundary. However, Rienstra [18, 37] noticed there also exist *surface modes* which are disturbances localized near the impedance boundary and strongly connected with the motion of the boundary. For any given frequency ω, surface modes are mostly easily noticeable in having unexpected axial wavenumbers k which do not correspond to an equivalent acoustic mode. This is illustrated in figure 2, which shows two cut-on acoustic modes, an infinite discrete number of evanescent acoustic modes, and (for uniform flow given by the $\times$ symbols) four surface modes. The position of surface modes can be predicted using a surface mode dispersion relation proposed by Rienstra [18] and generalized in [39]. For uniform flow in a cylinder the surface mode dispersion relation of [39] equivalent to (3) is

$$\sqrt{k^2 + m^2 - (\omega - Mk)^2} - \frac{(\omega - Mk)^2}{\mathrm{i}\omega Z} = 0 \qquad (7)$$

with $\mathrm{Re}(\sqrt{\cdots}) > 0$. For a locally reacting impedance, this allows up to four surface modes, as discussed in detail in [18, 39], and figure 2 is a case with all four present. Rienstra [18] went on to tentatively identify the surface mode in the upper-right k-plane quadrant as a hydrodynamic instability, although from §3 we now know that any instability analysis with slipping flow cannot be fully correct.

Using the wellposed boundary condition (6) leads to more complicated surface mode behaviour, with now up to six surface modes being possible (and five being shown for $+$ and $\circ$ in solutions in figure 2). One of the two possible surface modes in the upper-right k-plane quadrant turns out to indeed be a convective instability [38], although another surface mode in this quadrant is stable. Further investigation of this is ongoing.

Both Rienstra & Darau [30, 31] and Brambley [27, 32] find that for very thin boundary layers (i.e. very small δ) an absolute instability is present, while for thicker (though still relatively thin) boundary layers only convective instability is present; whether there is a stable regime is still an open question. As $\delta \to 0$ the absolute instability growth rate tends to infinity, recovering the illposed behaviour for $\delta = 0$. Very recently, Marx [40] showed that two different types of absolute instability may be present in a rectangular duct, depending on whether the surface mode on one wall is affected by the opposite wall or not.

In sheared flow, as well as modal instabilities mentioned above there may also exist nonmodal instabilities which grow algebraically rather than exponentially due to the critical layer [34–36]. For the case of linear shear in a cylindrical duct, [36] showed that the effect of the critical layer is negligible provided the fluid is not forced within the critical layer, with the critical layer itself producing an algebraic decay of $O(1/x^4)$. However, for nonlinear shear [35] suggests the decay may be only of $O(1/x)$, and indeed the mathematics of [36] are acknowledged to break down for nonlinear shear. Even so, neither predict algebraic growth, as occurs for swirling flow [e.g. 41].

5 Experiments and viscous effects

Experimental results suitable for validating any of the above are rather scarce, with experiments more often being used to back out values of the impedance Z for which numerical or mathematical models give the closest agreement with experiment (so called impedance eduction, e.g. [42, 43]). The existence of an instability has only recently been shown experimentally [21], although other previous experimental results do suggest instability [20, e.g.]. However, Renou and Aurégan [22] performed experiments suggesting that even fully accounting for sheared flow using (1) is not enough to accurately predict experimental results. Using an impedance boundary condition different from (2) inspired by accounting for viscosity within a boundary layer [44], they showed good agreement between experiment and the mathematical uniform flow solution. This strongly suggests viscosity is necessary within the impedance boundary model for accuracy.

Aurégan et al. [44] considered a thin viscous sheared flow over a rigid perforated surface. They found the inviscid impedance boundary condition (2) was attained

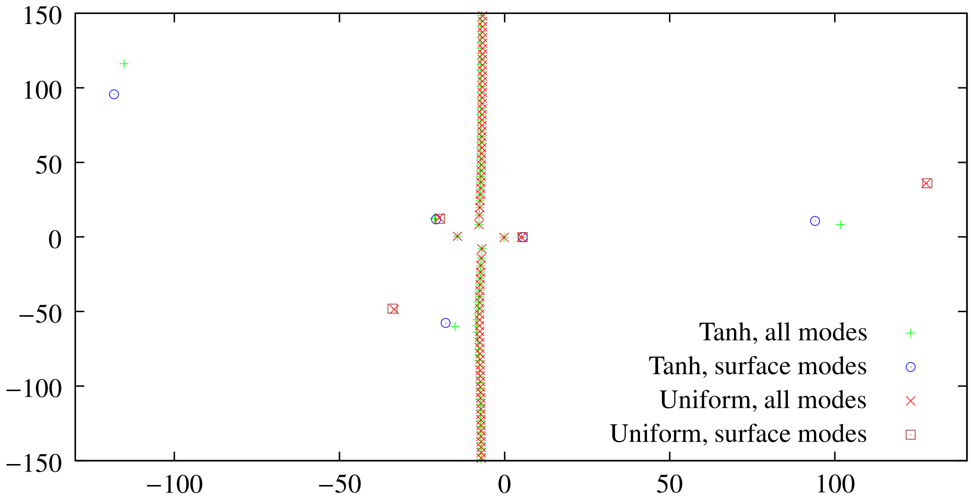

Figure 2: Comparison of axial wavenumbers (k) in the complex k-plane for (+) numerically-calculated solutions to (1) and (○) the surface mode dispersion relation of [38], both for $U(r) \approx M \tanh((1-r)/\delta)$, and (×) solutions to (3) and (□) the corresponding surface mode dispersion relation of [39], both for uniform flow with $U(r) \equiv M$. $\omega = 10$, $m = 5$, $M = 0.5$, $\delta = 10^{-3}$ and $Z = 1 - 2.5\mathrm{i}$, the same parameters as in figure 3 of [38].

only for high frequencies, while for low frequencies they predicted normal mass-flux to be constant across the boundary layer (equivalent to matching normal velocity in the absence of temperature gradients), while for mid frequencies the whole nature of the boundary condition was different. More recently, [45] extended [44] to avoid certain assumptions, and found a significant difference (at least in the low frequency limit) between a rigid permeable surface such as a rigid perforated sheet and a compliant impermeable surface such as a thin elastic sheet. Moreover, [45] showed that viscosity alone was not enough to yield a wellposed boundary condition, with presumably the next order terms in the boundary layer thickness being necessary for wellposedness, as in the inviscid case.

6 Conclusion

For slipping flow over an impedance boundary, the standard Myers, or Ingard–Myers, impedance boundary condition (2) assumes an infinitely-thin inviscid shear layer at the impedance boundary, leading to Kelvin–Helmholtz-type instabilities, difficulty with time domain numerical convergence and frequency domain causality, and a mathematically illposed problem. The problem is regularized by assuming the boundary layer is of width δ and accounting for the $O(\delta)$ correction terms, leading to a modified Myers boundary condition [27, 31]. Despite viscosity on its own not regularizing the problem [45], viscosity within the boundary layer appears essential to accurately predict experimental results [22]. Independently of the boundary condition used, a locally reacting impedance $Z(\omega)$ must satisfy Rienstra's four requirements to be physically realisable [24].

Using a wellposed modified Myers boundary condition, flow over an impedance boundary is predicted to be absolutely unstable for extremely thin boundary layers and only convectively unstable for thicker (though still relatively thin) boundary layers [27, 31], with the convective instability being one of the ones predicted by Rienstra [18, 38]. Such a boundary condition is expected to also regularize time domain numerical instabilities, although this is currently work in progress [46].

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