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Comment

Correction to “On the acoustics of an impedance liner with shear and cross flow”, by Campos, Legendre & Sambuc

E. J. Brambley[1]

[1]DAMTP, University of Cambridge, CMS, Wilberforce Road, Cambridge, CB2 1TA, UK

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Author for correspondence:
E. J. Brambley
e-mail:
E.J.Brambley@damtp.cam.ac.uk

Campos, Legendre, and Sambuc [1] derived an acoustic-vortical wave equation for the study of acoustics within a sheared flow $U_0(y)\boldsymbol{e_x}$ over an acoustic lining with a constant cross flow $V_0\boldsymbol{e_y}$ through the lining. Unfortunately, their derivation makes inconsistent assumptions, and the resulting wave equation is therefore incorrect. This comment points out the error, and derives a corresponding equation using the same approximations as [1].

The situation considered is as shown in figure 1(a). A mean flow $\boldsymbol{v_0} = U_0(y)\boldsymbol{e_x} + V_0\boldsymbol{e_y}$ flows across an acoustic liner located along $y = 0$. This mean flow should satisfy the inviscid governing equations of conservation of mass, momentum, and entropy,

$$\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v_0} = 0, \qquad (0.1a)$$

$$\rho_0\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\boldsymbol{v_0} + \boldsymbol{\nabla}p_0 = 0, \qquad (0.1b)$$

$$\boldsymbol{v_0}\cdot\boldsymbol{\nabla}p_0 - c_0^2\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0 = 0, \qquad (0.1c)$$

where $c_0^2 = \frac{\partial p_0}{\partial \rho_0}\Big|_{s_0}$ is the square of the (local) speed of sound. The given velocity $\boldsymbol{v_0}$ is solenoidal, $\boldsymbol{\nabla}\cdot\boldsymbol{v_0} = 0$, and hence conservation of mass (0.1*a*) gives $\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0 = 0$. The momentum equation (0.1*b*) implies that

$$\frac{\partial p_0}{\partial x} = -\rho_0 V_0\frac{\mathrm{d}U_0}{\mathrm{d}y}, \qquad \frac{\partial p_0}{\partial y} = 0, \qquad (0.2)$$

so that the entropy equation (0.1*c*) is not satisfied: $-\rho_0 U_0 V_0\frac{\mathrm{d}U_0}{\mathrm{d}y} \neq 0$ since U_0, V_0 and $\mathrm{d}U_0/\mathrm{d}y$ are all supposed nonzero. Therefore, the base flow assumed by Campos et al. [1] does not satisfy the governing equations. This leads to ambiguity in deriving a wave equation based on this mean flow, as will be seen below.

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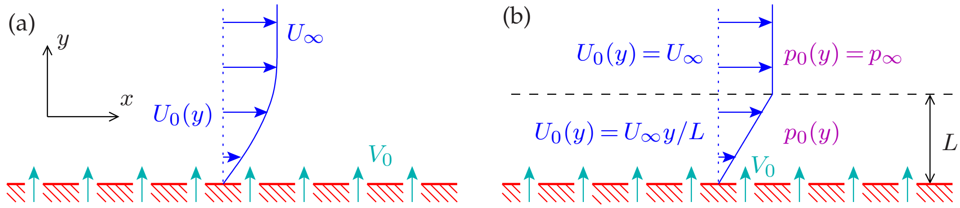

Figure 1. (a) The general situation considered both here and by Campos et al. [1]. The total mean flow is $\boldsymbol{v_0} = U_0(y)\boldsymbol{e_x} + V_0\boldsymbol{e_y}$. (b) The specialization considered by Campos et al. [1]. The mean pressure within the shear layer is $p_0(y) = p_\infty - \rho_0 V_0 U_\infty x/L$.

The expressions for the partial derivatives of p_0 given in (0.2) are in general incompatible, meaning that (assuming $\frac{1}{\rho_0}\frac{\mathrm{d}U_0}{\mathrm{d}y}$ is not everywhere constant) there is no function p_0 satisfying (0.2). In the linear shear case specifically considered by Campos et al. [1], shown in figure 1(b), $p_0 = p_\infty - \rho_0 V_0 U_\infty x/L$ for $y < L$ but $p_0 = p_\infty$ for $y > L$, leading to a mean pressure jump across $y = L$ for $x \neq 0$. In order to proceed, the approximation made by Campos et al. [1] is that

$$|x| \ll \frac{p_0(0)}{\rho_0 V_0 \mathrm{d}U_0/\mathrm{d}y}, \tag{0.3}$$

so that the variation of p_0 with x may be considered to be small, and therefore p_0 may be approximated as constant. While this is true for the *numerical value* of p_0, it is not true for the *derivative* $\partial p_0/\partial x$, which remains nonzero and potentially of significant magnitude. This is in effect a Boussinesq approximation where both p_0 and $\partial p_0/\partial x$ are considered constant and nonzero. Unfortunately, Campos et al. [1] assume both that $\boldsymbol{\nabla} p_0 = 0$ (in deriving their equation 2.18) and that $\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\boldsymbol{v_0} = V_0 \mathrm{d}U_0/\mathrm{d}y\boldsymbol{e_x}$ (in deriving their equation 2.21a), which are inconsistent with conservation of x-momentum (0.1*b* above). Their derived wave equation (2.25 of ref. 1) is therefore incorrect, as seen in §2 below.

Even with the Boussinesq assumption (0.3), the mean flow entropy equation (0.1*c*) remains unsatisfied. The mean flow assumed by Campos et al. [1] therefore requires a rather unphysical steady external cooling in order to be realized, as will be seen next.

1. A consistent (albeit artificial) mean flow

In order to have a consistent mean flow, and therefore a unique linearization of the governing equations about that mean flow, we will assume here a steady heat source Q_0. While this is certainly not the only possible consistent extension of [1], it is the simplest extension that allows the same velocity and density as Campos et al. [1, equations (2.5) and (2.11a)]. The full governing equations are then

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho}\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{v}} = 0, \qquad \tilde{\rho}\frac{D\tilde{\boldsymbol{v}}}{Dt} + \boldsymbol{\nabla}\tilde{p} = 0, \qquad \frac{D\tilde{p}}{Dt} - \tilde{c}^2\frac{D\tilde{\rho}}{Dt} = \frac{\tilde{c}^2\tilde{\beta}Q_0}{\tilde{c}_p}, \tag{1.1}$$

where $D/Dt = \partial/\partial t + \tilde{\boldsymbol{v}}\cdot\boldsymbol{\nabla}$ is the material derivative, and

$$\tilde{c}^2 = \left.\frac{\partial\tilde{p}}{\partial\tilde{\rho}}\right|_{\tilde{s}}, \qquad \tilde{\beta} = -\frac{1}{\tilde{\rho}}\left.\frac{\partial\tilde{\rho}}{\partial\tilde{T}}\right|_{\tilde{p}}, \qquad \tilde{c}_p = \tilde{T}\left.\frac{\partial\tilde{s}}{\partial\tilde{T}}\right|_{\tilde{p}} \tag{1.2}$$

are respectively the square of the speed of sound, the coefficient of thermal expansion, and the specific heat at constant pressure respectively. For a perfect gas, $\tilde{c}^2 = \gamma\tilde{p}/\tilde{\rho}$ and $\tilde{c}^2\tilde{\beta}/\tilde{c}_p = \gamma - 1$, where $\gamma = c_p/c_v$ is the ratio of specific heats.

Tildes here denote a total quantity, which is considered as a sum of a steady mean flow and a small unsteady perturbation, e.g. $\tilde{\rho} = \rho_0 + \rho'$. Substituting the mean flow assumptions above into these full governing equations (1.1) and assuming a Boussinesq approximation for p_0 gives the consistent solution $\boldsymbol{v_0} = U_0(y)\boldsymbol{e_x} + V_0\boldsymbol{e_y}$ with ρ_0 constant, p_0 approximated as constant, and

$$\nabla p_0 = -\rho_0 V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}\boldsymbol{e_x}, \qquad Q_0 = -\frac{\rho_0 c_{p0}}{\beta_0}\frac{U_0}{c_0}\frac{V_0}{c_0}\frac{\mathrm{d}U_0}{\mathrm{d}y}. \tag{1.3}$$

Hence, a rather unphysical external cooling $-Q_0$ is needed for this flow to exist in practice.

It should be noted that the sound speed $c_0^2 = c_0^2(p_0, \rho_0)$ has a nonzero gradient,

$$\nabla c_0^2 = -\alpha V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}\boldsymbol{e_x}, \qquad \text{where} \qquad \alpha = \rho_0 \left.\frac{\partial c_0^2}{\partial p_0}\right|_{\rho_0}, \tag{1.4}$$

(where for a perfect gas $\alpha = \gamma$), and thus we may not neglect gradients of c_0^2 in deriving our wave equation. Unfortunately, Campos et al. [1, equation (2.13)] did neglect gradients of c_0^2 in deriving their wave equation.

2. Derivation of the “wave equation” following ref. [1]

Linearizing the governing equations (1.1) about the steady mean flow (1.3) gives

$$\frac{D_0\rho'}{D_0 t} + \rho_0 \nabla\cdot\boldsymbol{v'} = 0, \tag{2.1a}$$

$$\rho_0\left(\frac{D_0\boldsymbol{v'}}{D_0 t} + v'\frac{\mathrm{d}U_0}{\mathrm{d}y}\boldsymbol{e_x}\right) + \underbrace{\rho' V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}\boldsymbol{e_x}}_{*} + \nabla p' = 0, \tag{2.1b}$$

$$\frac{D_0 p'}{D_0 t} \underbrace{- u'\rho_0 V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}}_{\dagger} - c_0^2 \frac{D_0\rho'}{D_0 t} = Q_0\left[p'\left.\frac{\partial}{\partial p_0}\right|_{\rho_0} + \rho'\left.\frac{\partial}{\partial \rho_0}\right|_{p_0}\right]\frac{c_0^2\beta_0}{c_{p0}} \equiv \mathcal{Q}_0, \tag{2.1c}$$

where $D_0/D_0t = \partial/\partial t + \boldsymbol{v_0}\cdot\nabla$ is the material derivative with respect to the mean flow, the velocity perturbation is $\boldsymbol{v'} = u'\boldsymbol{e_x} + v'\boldsymbol{e_y}$, and $\mathcal{Q}_0 \equiv 0$ for a perfect gas. The term marked † was erroneously omitted by Campos et al. [1], and originates from $\partial p_0/\partial x$ given in (0.2). Were it true that $\nabla p_0 \equiv 0$, as assumed by Campos et al. [1], then both the term marked $*$ in (2.1*b*) and the term marked † in (2.1*c*) would be identically zero. Campos et al. [1] included the term marked $*$ and excluded the term marked †, showing that their wave equation is inconsistently derived.

We now follow the procedure of Campos et al. [1]. Noting that

$$\nabla\cdot\frac{D_0\boldsymbol{v'}}{D_0 t} = \frac{D_0}{D_0 t}(\nabla\cdot\boldsymbol{v'}) + \frac{\mathrm{d}U_0}{\mathrm{d}y}\frac{\partial v'}{\partial x}, \tag{2.2}$$

using (2.1*a*) to eliminate $D_0\rho'/D_0t$ from (2.1*c*) and then taking $D_0/D_0t((2.1c)/c_0^2) - \nabla\cdot(2.1b)$ leads to

$$\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 p'}{D_0 t}\right) - \rho_0 V_0 \frac{D_0}{D_0 t}\left(\frac{u'}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) - 2\rho_0\frac{\mathrm{d}U_0}{\mathrm{d}y}\frac{\partial v'}{\partial x} - \frac{\partial\rho'}{\partial x}V_0\frac{\mathrm{d}U_0}{\mathrm{d}y} - \nabla^2 p' = \frac{D_0}{D_0 t}\frac{\mathcal{Q}_0}{c_0^2}, \tag{2.3}$$

which is the equivalent of (2.22) of ref. 1. Using the x-momentum perturbation equation (2.1*b*) to substitute for D_0u'/D_0t and then taking $D_0(2.3)/D_0t$ yields

$$\frac{D_0}{D_0 t}\left(\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 p'}{D_0 t}\right) - \nabla^2 p'\right) - 2\rho_0\frac{\partial}{\partial x}\frac{D_0}{D_0 t}\left(v'\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) - V_0\frac{\partial}{\partial x}\frac{D_0}{D_0 t}\left(\rho'\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)$$
$$+ V_0\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\left[\rho_0 v'\frac{\mathrm{d}U_0}{\mathrm{d}y} + \rho' V_0\frac{\mathrm{d}U_0}{\mathrm{d}y} + \frac{\partial p'}{\partial x}\right] - \rho_0 u'\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)\right) = \frac{D_0^2}{D_0 t^2}\frac{\mathcal{Q}_0}{c_0^2}. \tag{2.4}$$

After a significant amount of algebra, this becomes

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$$\frac{D_0}{D_0 t}\left(\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 p'}{D_0 t}\right) - \nabla^2 p'\right) + 2\frac{\mathrm{d}U_0}{\mathrm{d}y}\frac{\partial^2 p'}{\partial x \partial y} - (\alpha - 1)\frac{V_0^2}{c_0^4}\left(\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)^2 \frac{D_0 p'}{D_0 t}$$
$$+ 2V_0 \frac{\partial p'}{\partial x}\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) - \frac{V_0}{c_0^2}\left(\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)^2 \frac{\partial p'}{\partial y}$$
$$= V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}\left[\frac{V_0}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y} - \frac{\partial}{\partial x}\right]\left[u'\rho_0 \frac{V_0}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y} + \frac{\mathcal{Q}_0}{c_0^2}\right]$$
$$- (\rho_0 v' + \rho' V_0) V_0 \frac{\mathrm{d}U_0}{\mathrm{d}y}\left[\frac{V_0}{c_0^2}\frac{\mathrm{d}^2 U_0}{\mathrm{d}y^2} + 2\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)\right]$$
$$+ u'\rho_0 V_0 \frac{D_0^2}{D_0 t^2}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) + V_0^2 \frac{\mathrm{d}^2 U_0}{\mathrm{d}y^2}\frac{\partial \rho'}{\partial x} + 2\rho_0 V_0 \frac{\mathrm{d}^2 U_0}{\mathrm{d}y^2}\frac{\partial v'}{\partial x} + \frac{D_0^2}{D_0 t^2}\frac{\mathcal{Q}_0}{c_0^2}, \quad (2.5)$$

where

$$\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) = \frac{V_0}{c_0^2}\frac{\mathrm{d}^2 U_0}{\mathrm{d}y^2} + U_0 V_0 \frac{\alpha}{c_0^4}\left(\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)^2, \quad (2.6)$$

$$\frac{1}{V_0^2}\frac{D_0^2}{D_0 t^2}\left(\frac{1}{c_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}y}\right) = \frac{1}{c_0^2}\frac{\mathrm{d}^3 U_0}{\mathrm{d}y^3} + 3U_0 \frac{\alpha}{c_0^4}\frac{\mathrm{d}U_0}{\mathrm{d}y}\frac{\mathrm{d}^2 U_0}{\mathrm{d}y^2} + \frac{1}{c_0^4}\left(\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)^3 \left[\alpha + 2\alpha^2 \frac{U_0^2}{c_0^2} - \rho_0 U_0 \left.\frac{\partial \alpha}{\partial p_0}\right|_{\rho_0} \frac{D_0 p_0}{D_0 t}\right]. \quad (2.7)$$

This is the equivalent of (2.23) of ref. 1. Setting $V_0 = 0$ recovers[1] the Pridmore-Brown equation [2], while setting $U_0(y)$ to be constant recovers the convected wave equation. By assuming a linear shear $U_0(y) = \kappa y$ and a perfect gas, equation (2.5) "simplifies" to

$$\frac{D_0}{D_0 t}\left(\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 p'}{D_0 t}\right) - \nabla^2 p'\right) + 2\kappa \frac{\partial^2 p'}{\partial x \partial y} - (\gamma - 1)\kappa^2 \frac{V_0^2}{c_0^4}\frac{D_0 p'}{D_0 t} + 2\kappa^2 U_0 V_0^2 \frac{\gamma}{c_0^4}\frac{\partial p'}{\partial x} - \kappa^2 \frac{V_0}{c_0^2}\frac{\partial p'}{\partial y}$$
$$= -\kappa^2 \rho_0 \frac{V_0^2}{c_0^2}\frac{\partial u'}{\partial x} - 2\kappa^3 U_0 V_0^2 \frac{\gamma}{c_0^4}(\rho_0 v' + \rho' V_0) + \rho_0 \kappa^3 \frac{V_0^3}{c_0^4}\left(1 + 2\gamma^2 \frac{U_0^2}{c_0^2}\right) u', \quad (2.8)$$

which does not rearrange to the wave equation derived by Campos et al. [1, equation (2.25)]. We must therefore conclude that the wave equation ref. 1 is based on is erroneous.

It may, in certain cases, be possible to rearrange (2.5) or (2.8) into a wave equation in only one variable, say p'; however, doing so would yield an unhelpfully complicated equation of even higher order for this rather artificial mean flow, and is therefore not pursued further here.

3. A linear shear boundary layer

Campos et al. [1] considered a linear boundary layer of thickness L between the liner and a region of uniform flow, as shown in figure 1(b). As already commented, the mean flow pressure p_0 is discontinuous at $y = L$. Moreover, equation (2.5) includes second and third derivatives of $U_0(y)$, so that delta functions $\delta(y - L)$ and $\delta'(y - L)$ would be introduced in this case, which would cause p' to have a discontinuous derivative when crossing $y = L$. This flow profile is therefore not equivalent to solving a uniform flow $U_0(y) = U_\infty$ and an infinite linear shear $U_0(y) = \kappa y$ and matching the solutions together at $y = L$, as was done by Campos et al. [1, equation (3.19)].

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2 D. C. Pridmore-Brown. Sound propagation in a fluid flowing through an attenuating duct. *J. Fluid Mech.*, 4:393–406, 1958.

[1] Although note that V_0 multiplies the highest y-derivative of p' in (2.5), and so the limit $V_0 \to 0$ may be a singular limit.