NONLINEAR ACOUSTICS IN A GENERAL CURVED DUCT

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A method to solve for the weakly nonlinear acoustics in a general curved duct is presented. Pressure and longitudinal velocity are expressed as an infinite sum of spatial straight-duct modes and temporal Fourier harmonics. The pressure and longitudinal velocity are linked by the admittance, which is independent of the acoustic source and satisfies a first order differential equation which must be solved numerically from the outlet backwards towards the acoustic source. We introduce for the first time the novel concept of a nonlinear admittance correction, which also satisfies a first order differential equation similar to the linear admittance. The pressure is then given by solving a first order differential equation numerically from the acoustic source forwards towards the outlet. As an example, the method is applied to a two dimensional duct of constant width previously presented in literature. The results show that including nonlinearity has an important effect on waveform shape inside a curved duct.

Keywords: weak nonlinearity, curved waveguide, asymptotics

1. Introduction

Curved ducts arise in a wide variety of engineering applications, including aeroacoustics and musical acoustics. Previously, most worked on curved ducts has made linear approximations to the Euler equations [1] and hence the results are only valid for small amplitudes. As has been shown experimentally, many physical systems of complex geometry such as the resonator of a trombone feature acoustics of finite (though still small) amplitude where shock formation plays an important role [2, 3]. It is therefore important to gain an understanding of the nonlinear dynamics in the regions of curvature.

In section 2 we introduce a method based on the multimodal method of Félix and Pagneux [4, 5], extending their work into the weakly nonlinear regime. Working in a curvilinear coordinate system aligned with the duct, a multimodal method correct to second order in the wave amplitude is formulated for a general shaped duct via the inclusion of a ‘nonlinear admittance’ term. In section 3 we present the boundary conditions for an infinite straight duct outlet and briefly discuss their stability. In section 4 we introduce our numerical scheme, the exponential integrator method. In section 5 we present the results of an example duct previously published in literature [4] and show the importance of including the nonlinear terms. In section 6 we discuss further work for which there is insufficient room in the present article.

2. Governing Equations

We begin with the inviscid mass and momentum conservation equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p. \tag{1}$$

We linearize about a steady state of rest with density ρ_0 and sound speed c_0, giving $p = \rho_0 c_0^2(\hat{p}_0 + \hat{p}')$, $\rho = \rho_0(1+\hat{\rho}')$ and $\mathbf{u} = c_0\hat{\mathbf{u}}'$. The perturbations $\hat{p}' \sim \hat{\rho}' \sim \hat{\mathbf{u}}' \sim O(M)$ where $M < 1$ is the perturbation Mach number which is assumed small. Discarding terms of $\mathcal{O}(M^3)$ and higher gives

$$\frac{1}{c_0}\frac{\partial \hat{\rho}'}{\partial t} + \nabla\cdot\hat{\mathbf{u}}' = -\hat{\mathbf{u}}'\cdot\nabla\hat{\rho}' - \hat{\rho}'\nabla\cdot\hat{\mathbf{u}}', \qquad \frac{1}{c_0}\frac{\partial \mathbf{u}'}{\partial t} + \nabla\hat{p}' = -\hat{\mathbf{u}}'\cdot\nabla\hat{\mathbf{u}}' + \hat{\rho}'\nabla\hat{p}', \tag{2}$$

where the linear expression for $\frac{\partial \hat{\mathbf{u}}'}{\partial t} = -\nabla\hat{p}'$ has been used in the quadratic term on the RHS of the momentum equation. Expanding the equation of state at fixed entropy in a Taylor series gives

$$p' = c_0^2\rho' + \frac{B}{2A}\frac{c_0^2}{\rho_0}\rho'^2 + \cdots \qquad \text{with} \qquad A = \rho_0\left(\frac{\partial p}{\partial \rho}\right)_s = \rho_0 c_0^2, \qquad B = \rho_0^2\left(\frac{\partial^2 p}{\partial \rho^2}\right)_s. \tag{3}$$

Inverting (correct to second order) and using non-dimensional variables gives

$$\hat{\rho}' = \hat{p}' - \frac{B}{2A}\hat{p}'^2 \tag{4}$$

which can be used to eliminate the density from the mass and momentum equations,

$$\frac{1}{c_0}\frac{\partial \hat{p}'}{\partial t} + \nabla\cdot\hat{\mathbf{u}}' = -\hat{p}\nabla\cdot\hat{\mathbf{u}}' - \hat{\mathbf{u}}'\cdot\nabla\hat{p} + \frac{B}{2Ac_0}\frac{\partial}{\partial t}\left(\hat{p}'^2\right), \tag{5a}$$

$$\frac{1}{c_0}\frac{\partial \hat{\mathbf{u}}'}{\partial t} + \nabla\hat{p}' = -\hat{\mathbf{u}}'\cdot\nabla\hat{\mathbf{u}}' + \hat{p}'\nabla\hat{p}'. \tag{5b}$$

The pressure and velocity can now be expressed as Fourier series in multiples of the (monochromatic) acoustic source frequency ω (with upper indices used to denote temporal decompositions),

$$\hat{p} = \sum_{a=-\infty}^{\infty} P^a(\mathbf{x})e^{-ia\omega t}, \qquad \hat{\mathbf{u}} = \sum_{a=-\infty}^{\infty} \mathbf{U}^a(\mathbf{x})e^{-ia\omega t}, \tag{6}$$

where $P^{-a} = P^{a*}$ and $\mathbf{U}^{-a} = \mathbf{U}^{a*}$ so that both p and $\mathbf{u}$ are real and can be safely substituted into quadratic terms. Substituting (6) into (5) and equating terms of the Fourier series yields

$$-iakP^a + \nabla\cdot\mathbf{U}^a = \sum_{b=-\infty}^{\infty}\left(-P^{a-b}\nabla\cdot\mathbf{U}^b - \mathbf{U}^{a-b}\cdot\nabla P^b - \frac{B}{2A}iakP^bP^{a-b}\right), \tag{7a}$$

$$-iak\mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty}\left(-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b + P^{a-b}\nabla P^b\right), \tag{7b}$$

where $k = \omega/c_0$ is the wavenumber. These are the general form of the equations we wish to solve.

2.1 Duct Geometry

While the following is for a 2D duct of constant width, the procedure is readily generalizable to three dimensions using the appropriate coordinate system. We define our duct geometry by its centreline $\mathbf{q}(s)$ as a function of the longitudinal arc length s from the pressure source inlet. A general position vector in the duct is given by

$$\mathbf{x}(s,r) = \mathbf{q}(s) + r\mathbf{e}_r, \qquad \text{where} \qquad \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \mathbf{e}_s, \qquad \frac{\mathrm{d}\mathbf{e}_s}{\mathrm{d}s} = \kappa(s)\mathbf{e}_r, \qquad \frac{\mathrm{d}\mathbf{e}_r}{\mathrm{d}s} = -\kappa(s)\mathbf{e}_s, \tag{8}$$

and $-h \leq r \leq h$ defines the transverse position inside the duct of width $2h$. The corresponding Lamé coefficients are $h_s = 1 - \kappa r$ and $h_r = 1$. Using these, equations (7) can now be expanded in their coordinate specific forms resulting in three coupled equations for the Fourier modes of pressure P^a, longitudinal velocity U^a and transverse velocity V^a,

$$-iakP^a + \frac{1}{1-\kappa r}\frac{\partial U^a}{\partial s} + \frac{\partial V^a}{\partial r} - \frac{\kappa V^a}{1-\kappa r} \tag{9a}$$
$$= \sum_{b=-\infty}^{\infty}\left(-ibkP^{a-b}P^b - ibkU^{a-b}U^b - ibkV^{a-b}V^b - \frac{B}{2A}iakP^bP^{a-b}\right),$$

$$-iakU^a + \frac{1}{1-\kappa r}\frac{\partial P^a}{\partial s} = \sum_{b=-\infty}^{\infty}\left(-\frac{U^{a-b}}{1-\kappa r}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} + \frac{\kappa}{1-\kappa r}U^{a-b}V^b + ibkP^{a-b}U^b\right), \tag{9b}$$

$$-iakV^a + \frac{\partial P^a}{\partial r} = \sum_{b=-\infty}^{\infty}\left(\frac{-U^{a-b}}{1-\kappa r}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{\kappa}{1-\kappa r}U^{a-b}U^b + ibkP^{a-b}V^b\right). \tag{9c}$$

Using equation (9c) the transverse velocity can be eliminated from the other two equations (retaining terms only up to quadratic in M). From here, the temporal Fourier modes are expanded about a basis of spatial *straight* duct modes,

$$P^a = \sum_{p=0}^{\infty} P_p^a(s)\psi_p(r), \qquad\qquad U^a = \sum_{p=0}^{\infty} U_p^a(s)\psi_p(r), \tag{10}$$

where the ψ_p satisfy the straight duct eigenvalue problem and normalization condition

$$\frac{\mathrm{d}^2\psi_p}{\mathrm{d}r^2} + \lambda_p^2\psi_p = 0, \qquad\qquad \left.\frac{\mathrm{d}\psi_p}{\mathrm{d}r}\right|_{r=\pm h} = 0, \qquad\qquad \int_{-h}^{h}\psi_p\psi_q\,\mathrm{d}r = \delta_{pq}. \tag{11}$$

The straight duct eigenmodes in this case are given by $\psi_p(r) = C_p\cos\big(\lambda_p(r-h)\big)$ with $\lambda_p = p\pi/2h$ and $C_p = \big(h(1+\delta_{p0})\big)^{-1/2}$. Using orthogonality of ψ_p we obtain the pressure and longitudinal velocity vector equations

$$\mathbf{u}' + \mathsf{M}\mathbf{p} = \mathcal{A}[\mathbf{u},\mathbf{u}] + \mathcal{B}[\mathbf{p},\mathbf{p}], \qquad\qquad \mathbf{p}' - \mathsf{N}\mathbf{u} = \mathcal{C}[\mathbf{u},\mathbf{p}]. \tag{12}$$

Here, calligraphic letter denote 5th rank tensors, and matrix multiplication and square brackets denote the following operations:

$$(\mathsf{M}\mathbf{p})_p^a = \sum_{q=0}^{\infty}\mathsf{M}_{pq}^a P_q^a; \qquad\qquad (\mathcal{A}[\mathbf{x},\mathbf{y}])_p^a = \sum_{b=-\infty}^{\infty}\sum_{q,r=0}^{\infty}\mathcal{A}_{pqr}^{ab}X_q^{a-b}Y_r^b. \tag{13}$$

M, N, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ all have analytic expressions which can be found after some elementary algebra.

Due to the presence of evanescent modes, (12) cannot be integrated directly. Following the work of Félix and Pagneux [4], we define a relation between the pressure and velocity in terms of the admittance. When solving for pressure, it turns out to be easier to work with the admittance rather than the impedance. We introduce the relationship

$$\mathbf{u} = \mathsf{Y}\mathbf{p} + \mathcal{Y}[\mathbf{p},\mathbf{p}] \tag{14}$$

where $\mathsf{Y} = \mathsf{Y}(s)$ is the classical linear admittance ($\mathsf{Y} = \mathsf{Z}^{-1}$ where Z is the impedance matrix of Félix and Pagneux) and $\mathcal{Y} = \mathcal{Y}(s)$ is the second order nonlinear correction to the impedance. By differentiating (14), substituting in (12), using (14) to express velocity modes in terms of pressure modes, and equating orders of magnitude, we are left with two equations

$$\mathsf{Y}' + \mathsf{YNY} + \mathsf{M} = 0, \tag{15}$$

$$\mathcal{Y}'[\mathsf{I},\mathsf{I}] + \mathcal{Y}[\mathsf{NY},\mathsf{I}] + \mathcal{Y}[\mathsf{I},\mathsf{NY}] + \mathsf{YN}\mathcal{Y}[\mathsf{I},\mathsf{I}] + \mathsf{Y}\mathcal{C}[\mathsf{Y},\mathsf{I}] - \mathcal{A}[\mathsf{Y},\mathsf{Y}] - \mathcal{B}[\mathsf{I},\mathsf{I}] = 0. \tag{16}$$

These equations are solved backwards along the duct from the outlet — where an appropriate radiation condition boundary condition is imposed — to the inlet. Once both parts of the admittance are found throughout the duct, they can be used to replace the velocity modes for pressure modes in (12). The result is a numerically stable first order equation for the pressure which can be integrated from the inlet to the outlet using suitable initial conditions at the inlet to describe the waveform there,

$$\mathbf{p}' = \mathsf{NY}\mathbf{p} + \mathsf{N}\mathcal{Y}[\mathbf{p},\mathbf{p}] + \mathcal{C}[\mathsf{Y}\mathbf{p},\mathbf{p}]. \tag{17}$$

Due to the varying admittance this equation takes into account a superposition of forward and backward travelling modes as determined by the radiation condition at the outlet, as well as their nonlinear interaction.

3. Boundary Conditions for an Infinite Straight Duct Outlet

The simplest outlet boundary condition for the admittance is an infinitely long straight duct for which we have only outgoing propagating waves and decaying evanescent waves. The generalization to infinite curved outlets is omitted here for the sake of brevity. As in an infinite straight duct no point can be distinguished from another longitudinally, we must have the admittance being a fixed point of the governing equations when $\kappa = 0$. Equation (15) becomes

$$\sum_{s=0}^{\infty} \left(\mathsf{Y}^a_{ps}\right)_{\text{fixed}} \left(\mathsf{Y}^a_{sq}\right)_{\text{fixed}} = \left(1 - \frac{\lambda_q^2}{a^2k^2}\right)\delta_{pq} \qquad \Rightarrow \qquad \left(\mathsf{Y}^a_{pq}\right)_{\text{fixed}} = \delta_{pq}\frac{k_p^a}{ak}, \tag{18}$$

where the k_p^a are the longitudinal wavenumbers of the duct modes. Roots should be chosen such that we have outgoing waves and decaying evanescent modes

$$k_p^a = \begin{cases} \sqrt{a^2k^2 - \lambda_p^2} & a > 0 \\ -\sqrt{a^2k^2 - \lambda_p^2} & a < 0 \\ +i\sqrt{\lambda_p^2 - a^2k^2} & \text{evanescent modes} \end{cases} \tag{19}$$

We also have the relation $\mathsf{Y}^{-a}_{pq} = \left(\mathsf{Y}^a_{pq}\right)^*$ at all points along the duct. Substituting (18) into (16), we get the fixed point of the nonlinear part of the admittance

$$\left(\mathcal{Y}^{ab}_{pqr}\right)_{\text{fixed}} = \frac{a\mathcal{A}^{ab}_{pqr}k_q^{a-b}k_r^b + k^2a(a-b)b\mathcal{B}^{ab}_{pqr} - b\mathcal{C}^{ab}_{pqr}k_p^a k_q^{a-b}}{k^2a(a-b)b(ik_p^a + ik_q^{a-b} + ik_r^b)}. \tag{20}$$

The stability of these fixed points can be demonstrated by considering small perturbations

$$\mathsf{Y}^a_{pq} = \delta_{pq}\frac{k_p^a}{ak} + \varepsilon^a_{pq}, \qquad \mathcal{Y}^{ab}_{pqr} = \left(\mathcal{Y}^{ab}_{pqr}\right)_{\text{fixed}} + \varepsilon^{ab}_{pqr}. \tag{21}$$

Substituting these into (15) and (16) respectively, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}s}\varepsilon^a_{pq} = -\left(ik_p^a + ik_q^a\right)\varepsilon^a_{pq} + \mathcal{O}(\varepsilon^2), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\varepsilon^{ab}_{pqr} = -\left(ik_p^a + ik_q^{a-b} + ik_r^b\right)\varepsilon^{ab}_{pqr}. \tag{22}$$

Hence we have stability provided $\mathrm{Im}(k_p^a) > 0$ for all modes — i.e. all evanescent modes decay (note that we are solving *backwards* along the duct, so a positive exponential corresponds to decay).

4. Numerical Method

The standard fixed step explicit RK4 method is used to solve equations (15) and (16). As equation (17) has multiple scales, solving directly by RK4 can cause numerical instabilities. To proceed, we write (17) in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{p} = \mathsf{NY}\mathbf{p} + \mathcal{N} \tag{23}$$

where we have split the RHS into linear and nonlinear parts. This has the *exact* solution

$$\mathbf{p}(s) = \exp\left\{\int_0^s \mathsf{NY}(t)\,\mathrm{d}t\right\}\mathbf{p}(0) + \int_0^s \exp\left\{\int_t^s \mathsf{NY}(u)\,\mathrm{d}u\right\}\mathcal{N}(t)\,\mathrm{d}t \tag{24}$$

If we consider a small step δs and approximate both NY and $\mathcal{N}$ as constant over that interval (NY is *exactly* constant in the case of a straight duct), we obtain a numerical stepping algorithm

$$\mathbf{p}(s+\delta s) = e^{\delta s \mathsf{NY}(s)} P_q^a(s) - (\mathsf{NY})^{-1}\left(\mathsf{I} - e^{\delta s \mathsf{NY}(s)}\right)\mathcal{N}(s) \tag{25}$$

This is a first order exponential integrator algorithm.

As we are truncating our series at a finite number of modes, energy begins to pool at the point of truncation leading to overestimations and eventually numerical instabilities. To compensate for this we introduce a numerical viscosity of the form

$$\frac{\mathrm{d}}{\mathrm{d}s}P^a \mapsto \frac{\mathrm{d}}{\mathrm{d}s}P^a - \frac{\alpha k a^2}{a_{\text{max}}}P^a \tag{26}$$

where α is small such that the computations remain stable. A similar approach is used by Fernando et al. [6] in their work.

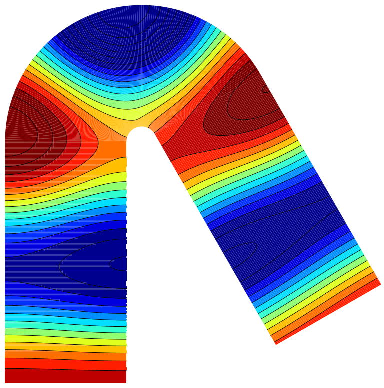
5. Results

We now apply this method to the same duct bend previously studied by Félix and Pagneux [4] with their linear multimodal method, and experimentally and with finite differences by Cabelli [7]. The duct consists of a circular bend of curvature $2h\kappa = \frac{8}{5}$ and length $1.6375 \times 2h$ with a straight inlet of length $4h$ and an infinite straight outlet. A sinusoidal plane piston source of frequency $2hk = 3$ and amplitude M is placed at the inlet so that $P_p^a = M\delta^{a1}\delta_{p0}$.

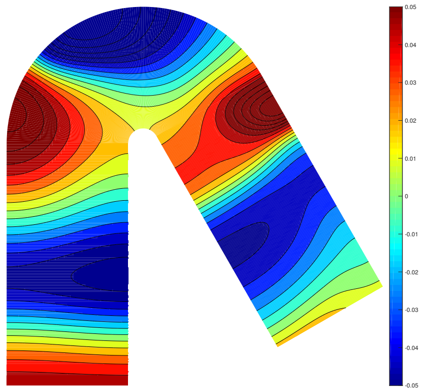
In the case when only a single temporal mode is taken, our method reduces to the Multimodal method of Félix and Pagneux [4] and the results agree perfectly. Figure 1a shows a contour plot of the pressure for the linear problem with 15 spatial modes being taken. Figures 1b to 1d show the changes to the solution when nonlinear effects are taken into account, with pressure disturbances of Mach numbers $M = 0.05$, $M = 0.10$ and $M = 0.15$ corresponding to shock formation distances in the outlet, in the bend and in the inlet respectively. 10 spatial modes and 10 temporal modes were taken. The plots show that for even modest Mach numbers wave steepening occurs within the bend of the duct, with high pressure regions travelling faster than low pressure regions - eventually forming a shock on the outside of the bend for higher Mach numbers.

6. Conclusion and Further Work

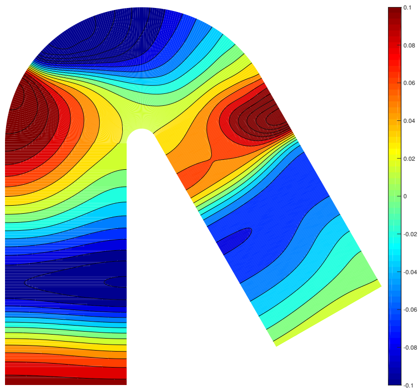
We have presented a semi-analytic method for deriving the weakly nonlinear propagation of waves in an arbitrarily curved 2D waveguide. The method extends the technique of Félix and Pagneux [4, 5] by introducing a nonlinear admittance correction. This method accounts for all weakly nonlinear effects (such as wave steepening), as well as reflection by the varying geometry. While the analysis given here is for a waveguide of constant width, the same analysis is valid for varying width waveguides, as well as for 3D ducts of arbitrary cross-sectional shape. Such shapes have applications in the



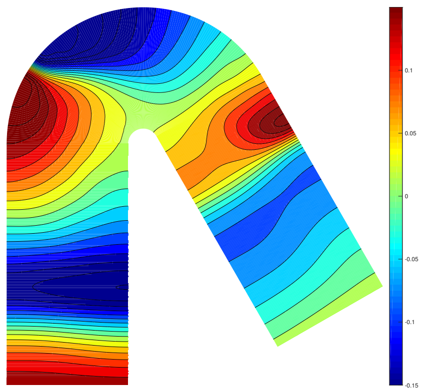
(a) Linear



(b) $M = 0.05$



(c) $M = 0.10$



(d) $M = 0.15$

Figure 1: Contour plots of pressure inside the duct bend

design of musical instruments, and for the prediction of noise from convoluted aircraft engine intakes. The latter have a significant mean flow that has not been included in the analysis presented here, but which could be incorporated.

While the amplitudes of these waves, which correspond to $171\,\mathrm{dB}$ ($M = 0.05$) to $180\,\mathrm{dB}$ ($M = 0.15$), are high when compared with room acoustics, they are of the correct order of magnitude for waves within musical instruments; for example, it is known that a trombone played fortissimo has a shock formation distance within the instrument [2, 3], while the pressure oscillations within the intakes and exhaust of aircraft engines are also of this order of magnitude.

Felix and Pagneux [4] use their multimodal method to predict the reflection properties of a curved duct. We can extend their work by introducing a "nonlinear reflection" term of the form

$$\mathbf{p}^- = \mathsf{R}\mathbf{p}^+ + \mathcal{R}[\mathbf{p}^+, \mathbf{p}^+] \tag{27}$$

relating forwards and backwards travelling waves. Using this, we are able to show the reflection properties of a duct depend not only on the frequency of the incident waveform but also its amplitude.

The admittance at the end of the duct was considered here to be that of an infinite straight waveguide. Any appropriate final admittance could have been used, including that of an open or closed end. An open ended waveguide has obvious applications to musical instruments. While the admittance, or equivalently the impedance, of an open end is well known, the nonlinear correction to that admittance is not, and further work on this nonlinear correction is ongoing.

At high amplitudes, viscous friction on the duct walls can become important [3, 6]. While it is possible that this could be introduced into the current method, possibly by the use of a fractional derivative, this has not been attempted here.

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