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# Nonlinear Acoustics in a Non-Parallel Boundary Layer over an Acoustic Lining

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**Sound within aircraft engines can be 140dB–160dB, and may be amplified by 1000× within a visco-thermal boundary layer over an acoustic lining, triggering nonlinear effects. It has been suggested that non-parallel effects could also be important, however acoustic linings give rise to different asymptotic scalings than most previous non-parallel work for hard walls. This paper presents an investigation into the effects of nonlinearity on the acoustics within a non-parallel boundary layer flow over an acoustic lining in a duct. The analysis combines the effects of shear, viscosity, and nonlinearity and uses a three-layer formulation to obtain analytic solutions. Unlike the parallel flow acoustics the non-parallel acoustics do not admit a highly oscillatory amplified acoustic streaming (zero frequency) solution. However there is still an amplified nonlinear solution that propagates out into the rest of the duct.**

## I. Introduction

Acoustic liners are an essential part of civilian aircraft engines, enabling them to meet ever stricter noise requirements. Sound within aircraft engines is loud, potentially 140dB-160dB, pushing the validity of the usually assumed linearised sound over a steady background flow. However, even if the sound within the engine ducting may be considered linear, an amplification mechanism by a factor of $1/\delta$, where $\delta$ is the boundary layer thickness (typically $\delta = 10^{-3}$), exists within a thin visco-thermal boundary layer[1]. Experimental evidence also suggests nonlinearity becomes important at lower amplitudes than expected for flow over an acoustic lining[2]. It has been shown that nonlinearity can cause unexpected acoustic streaming phenomena[3], although it has also been suggested that this may be an artefact of the assumption of parallel mean flow[4]. The purpose of this paper is to investigate this by considering weakly nonlinear acoustics in a developing non-parallel visco-thermal boundary layer over an acoustic lining.

Much work on acoustics in flow over acoustic linings uses the Myers boundary condition[5]. This matches the normal fluid displacements at the boundary, and comes from assuming the fluid is inviscid with an infinitely thin boundary layer. However this boundary condition gives a vortex sheet at the boundary and can be shown to be ill-posed[6, 7]. More recent work[8] gave a modified Myers boundary condition which took account of the shear in the boundary layer of the background flow but still ignored the effect of viscosity. This gave a closed form solution involving integrals over the mean flow boundary layer profile. However[9] the effect of viscosity on the acoustics occurs at the same order of magnitude as shear, and viscosity must also be taken into account. This means considering a finite thickness viscous boundary layer near the acoustic lining and solving for the acoustics in the boundary layer and then matching to an outer solution, which is assumed to act as an inviscid fluid with uniform flow[10, 11, 12]. This approach agrees more closely to results from solving the linearised Navier Stokes equations for the entire duct. We also showed last year[3] that non-linearity could give rise to unexpectedly large acoustic streaming modes which propagate out into the centre of the duct, suggesting that the effect of nonlinearity should also be considered.

All of the above work for an acoustic lining was done under the assumption of a parallel mean flow. It has been suggested[4] that the non parallel effects of developing boundary layers may have an important effect on the acoustics. However this work considered only hard wall boundary conditions which as we will show have a different scaling regime than acoustics in boundary layers near impedance linings.

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| | | | |
|---|---|---|---|
| Density | $\rho^* = \rho_0^*\rho$ | Pressure | $p^* = c_0^{*2}\rho_0^* p$ |
| Velocity | $\mathbf{u}^* = c_0^*\mathbf{u}$ | Viscosity | $\mu = c_0^* l^* \rho_0^* \mu$ |
| Distance | $x^* = l^* x$ | Thermal Conductivity | $\kappa^* = c_0^* l^* \rho_0^* c_p^* \kappa$ |
| Time | $t^* = l^*/c_0^* t$ | Temperature | $T^* = c_0^{*2}/c_p^* T$ |

Table 1: Dimensional and non-dimensional variables, where $*$ denotes a dimensional variable, with lengthscale $l^*$, velocity $c_0^*$, density $\rho_0^*$ and specific heat at constant pressure $c_p^*$.

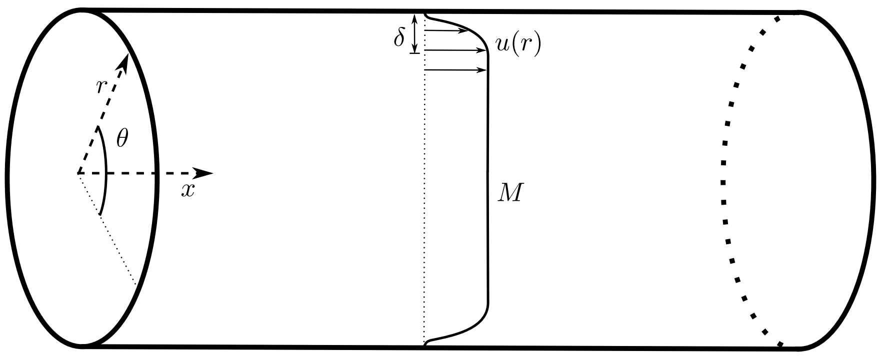



Figure 1: Diagram of the duct.

Here we consider the effect of nonlinearity with a non-parallel boundary layer mean flow. We restrict ourselves to the weakly nonlinear case $\varepsilon \ll \delta \ll 1$, where $\varepsilon$ is the acoustic amplitude $\tilde{p}/p_0$ and $\delta$ is a measure of the boundary layer thickness. We will also look sufficiently far downstream so that $1/k \ll x$ where $k$ is the streamwise wavenumber, which means that the boundary layer is sufficiently well developed that the effect of viscosity on the acoustics is restricted to an inner-inner region. This means that an expansion in $\varepsilon$ may be used with a three-layer formulation that can be solved analytically using asymptotic matching and a WKBJ solution.

## II. Mathematical Formulation

We consider the acoustics in a compressible viscous perfect gas inside a straight cylindrical duct. To begin, we non-dimensionalise all quantities as shown in table 1. The governing equations are then (Landau & Lifshitz)[13]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1a}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \sigma \tag{1b}$$

$$\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left( \mu^B - \tfrac{2}{3}\mu \right) \delta_{ij} \nabla \cdot \mathbf{u} \tag{1c}$$

$$\rho \frac{DT}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (\kappa \nabla T) + \sigma_{ij} \frac{\partial u_i}{\partial x_j} \tag{1d}$$

$$T = \frac{p}{(\gamma - 1)\rho} \tag{1e}$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ and $\gamma = c_p^*/c_v^*$ is the ratio of specific heats. We assume that the viscosities and thermal conductivity depend linearly on the temperature and are independent of pressure (Prangsma, Alberga & Beenakker)[14], so that we can write:

$$\mu = \frac{T}{T_0 \mathrm{Re}}, \qquad \mu^B = \frac{T}{T_0 \mathrm{Re}} \frac{\mu_0^{B^*}}{\mu_0^*}, \qquad \kappa = \frac{T}{T_0 \mathrm{Pr} \mathrm{Re}} \tag{2}$$

Where Re$= c_0^* l^* \rho_0^* / \mu_0^*$ is the Reynolds number, defined with respect to the sound speed, and Pr$= \mu_0^* c_p^* / \kappa_0^*$ is the Prandtl number. In the duct we use cylindrical coordinates $(r^*, \theta, x^*)$, where $l^*$ is now taken to be the radius of the duct, so that in non-dimensionalised variables the wall of the duct is at $r = 1$.

We take the reference values $\rho_0^*$, $T_0^*$, $\mu_0^*$, $\mu_0^{B^*}$ and $\kappa_0^*$ to be those of the centreline mean flow which is assumed to be uniform. This gives $T_0 = 1/(\gamma-1)$, $p_0 = 1/\gamma$ and the non-dimensionalised mean flow velocity $U_0 = M$ is the Mach number. We will consider the acoustics due to a source at $(r_0, x_0)$, sufficiently far from the leading edge so that we can ignore its effect and don't have to know anything about the geometry of the duct inlet.

We will assume that the acoustic lining reacts locally and can be modelled by a linear impedance relation $\tilde{p} = Z(\omega)\tilde{v}$ where $\tilde{p}$ is the acoustic pressure, $\tilde{v}$ the acoustic normal velocity and $\omega$ the frequency. Here we will use the mass-spring-damper impedance:

$$Z(\omega) = R - i\frac{b}{\omega} + i\omega d \tag{3}$$

This boundary condition results in there being non-zero acoustic normal velocity at the wall which then interacts with the large shear to amplify the acoustics within the boundary layer. This causes the asymptotic scalings to be different to the case of a hard wall, $Z = \infty$, where the acoustic normal velocity is zero at the wall.

### A. Three Layer Setup-Asymptotic model

We now consider a developing boundary layer flow near the wall of the duct, and we choose $x = 0$ to be the leading edge of the boundary layer. Using the standard Blasius boundary layer scaling of inertia with viscosity we have our boundary layer thickness $\delta_L$.

$$\delta_L = \sqrt{\frac{x}{M\mathrm{Re}}} = \delta\sqrt{\frac{x}{M}}$$

where we have defined $\delta^2 = 1/\mathrm{Re} \ll 1$. For the boundary layer approximation to be valid we need the streamwise lengthscale to be much larger than the radial lengthscale of the mean flow. That is $x_0 \gg \delta_L$, where $x_0$ is our distance downstream from the leading edge. This gives the requirement $x_0 \gg \frac{\delta^2}{M}$. Here we will use the compressible Blasius boundary layer profile which is an exact solution of the boundary layer equations for a flat plate with no pressure gradient and linear dependence of viscosity with the temperature.

We now consider the scalings of the acoustics. We have the time dependence of the acoustics given by the angular frequency $\Re(\omega)$ and we let $\lambda$ be the radial lengthscale from the wall over which the acoustics are affected by viscosity. Now we can consider the balance of the time dependent terms with the viscous terms:

$$\frac{\partial \tilde{u}}{\partial t} \sim \nu \tilde{u}_{yy} \quad \implies \quad \Re(\omega) \sim \frac{1}{\mathrm{Re}\lambda^2}$$

$$\lambda \sim \delta_L\sqrt{\frac{M}{\Re(\omega)x}} \sim \delta_L\sqrt{\frac{M}{k_r x}}$$

where $k_r = \Re(k)$ is the streamwise acoustic wavenumber. Now to avoid having to consider scattering off the leading edge we assume that the streamwise wavelength is shorter than our distance downstream. That is:

$$\frac{1}{k_r} \ll x_0$$

which gives us that $\lambda \ll \delta_L$ so we can introduce an inner-inner region of lengthscale $\lambda$.

We now have three distinct regions as shown in Figure 2 where different physical scalings hold.

In region III, the base flow is assumed to be approximately uniform and gradients of the acoustics are assumed to be $\mathcal{O}(1)$ so that the effect of viscosity is negligible at leading order. This means that the acoustics can be treated as being in an inviscid uniform flow, for which the solutions can be found to be in the form of Bessel functions.

In region II, the mean flow varies over a lengthscale $\delta_L$, gradients in the acoustics are assumed to be at most $\mathcal{O}(1/\delta_L)$, so the effect of viscosity is still negligible at leading order and the acoustics and be treated as being in a sheared inviscid flow.

In region I, the mean flow is approximately linear, but gradients in the acoustics are assumed to be $\mathcal{O}(1/\lambda)$ so the effect of viscosity is now important. Here we can treat the acoustics as being in a linearly sheared viscous flow.

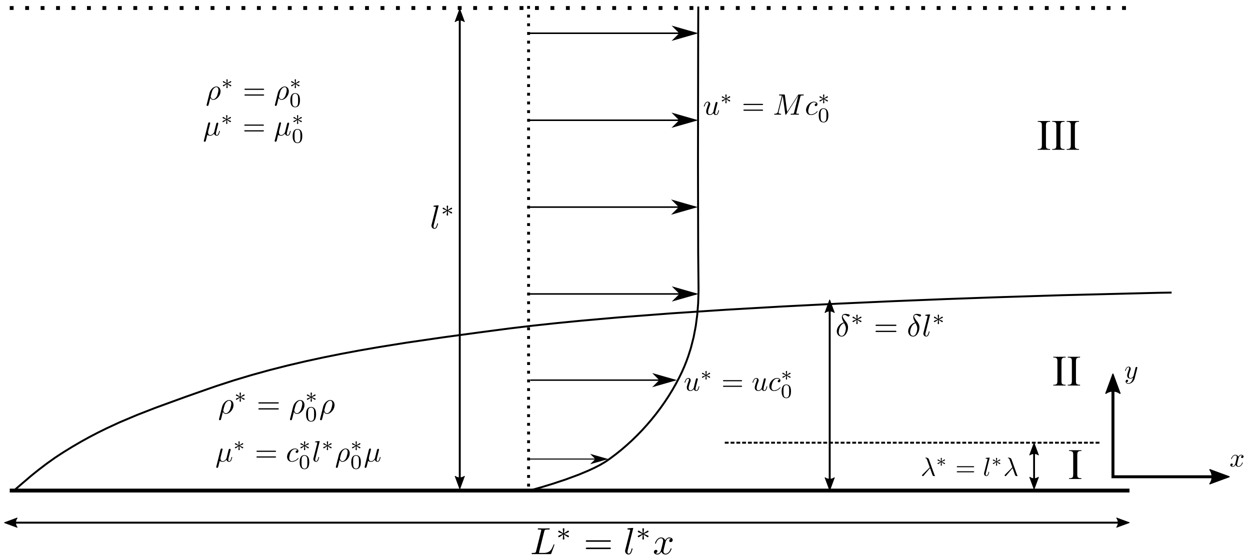



Figure 2: Diagram of the duct with inner acoustic boundary layer region

### B. Linear Acoustics

For the linear acoustics we Fourier transform in $x$, Laplace transform in time and take a Fourier series in $\theta$ so that we may consider only a single mode. We want to include the effect of the $x$-dependence of the mean flow so we will introduce a slow streamwise variable $\varepsilon_k x$, where $\varepsilon_k = \frac{M}{k_r x_0} \ll 1$ is the ratio between the wavelength and downstream distance. If we now expand our equations in each region to leading order in $\delta_L$, $\varepsilon_k$ and $\varepsilon$, where $\varepsilon \ll 1$ is the scale of the amplitude of the acoustic perturbations, we get a solution for the pressure perturbation of the form:

$$p' = \varepsilon \tilde{p}_0(r, \varepsilon_k x) \mathrm{e}^{i\omega t - i\int k(\varepsilon_k x)\mathrm{d}x - im\theta} + \mathcal{O}(\varepsilon_k, \delta_L, \varepsilon^2)$$

We can then use this ansatz to solve in each region. In III all quantities are $\mathcal{O}(\varepsilon)$:

$$\tilde{p}_0 = A(\varepsilon_k x) J_m(\alpha r) \qquad \tilde{u}_0 = \frac{k}{\omega - Mk} A(\varepsilon_k x) J_m(\alpha r) \qquad \tilde{v}_0 = \frac{-\alpha}{i(\omega - Mk)} A(\varepsilon_k x) J_m'(\alpha r)$$
$$\tilde{T}_0 = A(\varepsilon_k x) J_m(\alpha r) \qquad \tilde{w}_0 = \frac{m}{r(\omega - Mk)} A(\varepsilon_k x) J_m(\alpha r)$$

where $\alpha^2 = (\omega - Mk)^2 - k^2$ and $A(\varepsilon_k x)$ is a slowly varying function of $x$, which we will not need to solve for here.

In region II we transform into the mean flow boundary layer using the transformation $r = 1 - \delta_L \zeta$. This means that the mean flow terms depend only on the similarity variable $\zeta$ and not on $x$ and we find that $\tilde{u}$ and $\tilde{T} \sim \mathcal{O}(\varepsilon/\delta_L)$ and $\tilde{p}$, $\tilde{v}$ and $\tilde{w} \sim \mathcal{O}(\varepsilon)$,

$$\tilde{p}_0 = \text{const}, \qquad \tilde{v}_0 = -iC(\varepsilon_k x)(\omega - Uk), \qquad \tilde{u}_0 = -C(\varepsilon_k x) U_\zeta / \delta_L, \qquad \tilde{T}_0 = -C(\varepsilon_k x) T_\zeta / \delta_L.$$

Matching with the solution in region III, we find $A = -\frac{(\omega - Mk)^2}{\alpha J_m'(\alpha)} C$ and $\tilde{p}_0 = A J_m(\alpha)$.

In Region I we transform into the acoustic boundary layer using the transformation $r = 1 - \lambda y$. Now $\lambda$ is independent of $x$, and we can expand the mean flow quantities about their values at the wall, e.g. $U = 0 + \zeta U_\zeta(0) + ... = y\sqrt{\frac{M}{k_r x}} U_\zeta(0) + ...$, noting that $k_r x / M = \frac{x}{\varepsilon_k x_0} \sim 1/\varepsilon_k$. We find that $\tilde{u}_0 \sim \varepsilon/\delta_L$, $\tilde{T}_0 \sim \varepsilon (\varepsilon_k)^{1/2}/\delta_L$ and $\tilde{p}_0, \tilde{v}_0 \sim \varepsilon$. We then get the following solutions for $\tilde{u}_0$ and $\tilde{v}_0$:

$$\tilde{p}_0 = \text{const} \qquad \tilde{v}_0 = -i\omega C(\varepsilon_k x) \qquad \tilde{u}_0 = -U_\zeta(0) C(\varepsilon_k x) \left[ 1 - \exp\left( \frac{-y\sqrt{i\omega}}{\sqrt{k_r}(\gamma - 1)T(0)} \right) \right] / \delta_L$$

and $\tilde{u}_1 \sim \varepsilon(\varepsilon_k)^{1/2}/\delta_L$, $\tilde{v}_1 \sim \varepsilon(\varepsilon_k)^{1/2}$

$$\tilde{v}_1 = ikU_\zeta(0)C(\varepsilon_k x)\left[y\sqrt{\frac{\varepsilon_k x_0}{x}} + (\gamma-1)T(0)\sqrt{\frac{M}{i\omega x}}\exp\Big(\frac{-y\sqrt{i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}\Big)\right]$$
$$\tilde{u}_1 = -C(\varepsilon_k x)U_{\zeta\zeta}(0)\frac{y}{\delta_L}\sqrt{\frac{\varepsilon_k x_0}{x}} + \frac{U_\zeta(0)^2 C(\varepsilon_k x)}{4\delta_L\sqrt{x}}\left[\frac{ik\sqrt{\varepsilon_k x_0}}{\sqrt{i\omega k_r}(\gamma-1)T(0)}y^2 + \frac{3k\sqrt{\varepsilon_k x_0}}{\omega}y\right]\mathrm{e}^{\frac{-y\sqrt{i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}}$$

To find the modes $k^*(\omega)$ we apply the boundary condition $\tilde{p} = Z(\omega)\tilde{v}$ and match through each region to get the following dispersion relation at leading order:

$$\frac{Zi\omega\alpha}{(\omega-Mk)^2}J'_{\bar{m}}(\bar{\alpha}) = J_{\bar{m}}(\bar{\alpha})\left[1 - \frac{kU_\zeta(0)(\gamma-1)T(0)}{\omega}\sqrt{\frac{M}{i\omega x}}\right]^{-1} + \mathcal{O}(\varepsilon_k, \delta_L)$$

We could then find the next $\mathcal{O}(\varepsilon_k)$ solutions and apply a compatibility condition to find a differential equation for the slow streamwise variation $A(x)$. Here we will consider $A$ as being constant which is accurate to $\mathcal{O}(\varepsilon_k)$.

Previous work on a three layer model[11] used the assumption $\xi\delta_L^3 \sim 1/Re$, with $\xi \sim \mathcal{O}(1)$. This work assumed the flow was parallel, however if we set $\xi = \frac{M}{x\delta_L}$ we have the same scaling regime as in this paper and we find that at this order the dispersion relation above agrees with that previously found. At higher orders the corrections to the dispersion relation will not agree as the assumption of parallel flow means this previous work only gives a local solution.

### C. Forcing

We will consider a point mass source at $(r_0, 0, x_0)$ which turns on at $t = 0$. i.e.

$$q(t) = Re[q\delta(r-r_0)\delta(x-x_0)\frac{\delta(\theta)}{r_0}H(t)e^{i\omega_f t}]$$

where $H(t)$ is a Heaviside function. This then gives us our constant $A$ for the linear solution and hence $\tilde{p}_0$:

$$\tilde{p}_0 = \frac{e^{ikx_0}(\omega-Mk)}{4}\left(\frac{q}{\omega-\omega_f} + \frac{q^*}{\omega+\omega_f^*}\right)\left(\frac{(\omega-Mk)Y_m(\alpha) - i\alpha Y'_m(\alpha)Z_{\text{eff}}}{(\omega-Mk)J_m(\alpha) - i\alpha J'_m(\alpha)Z_{\text{eff}}}\right)J_m(\alpha r_0)J_m(\alpha r)$$

where $Z_{\text{eff}} = \frac{Z(\omega)\omega}{(\omega-Mk)}\left[1 - \frac{kU_\zeta(0)(\gamma-1)T(0)}{\omega}\sqrt{\frac{M}{i\omega x}}\right]$ is the effective impedance and $k_r$ now scales as $\mathcal{O}(\omega_f/M)$.

To find $p(x,t)$ it is necessary to invert the Fourier-Laplace transforms. Inverting the Fourier $x$ transformation is relatively simple, all the poles arise where $(\omega-Mk)J_m(\alpha) - i\alpha J'_m(\alpha)Z_{\text{eff}} = 0$ (i.e. modes of a duct with impedance $Z_{\text{eff}}$) and can be found numerically. The singularities at $\alpha = 0$ due to the $Y_m$ terms cancel out so do not need to be considered and there is no continuous spectrum. We can then use Jordan's lemma, closing in the upper half plane for upstream modes $x - x_0 < 0$ and in the lower half plane for downstream modes $x - x_0 > 0$, and get a sum over the poles.

$$\tilde{p}_0(\omega, x) = \pm i\sum_{k_*\in\mathcal{K}_\pm} Pe^{-ik_*(x-x_0)}\lim_{k\to k_*}(k-k_*)\left(\frac{(\omega-Mk)Y_m(\alpha) - i\alpha Y'_m(\alpha)Z_{\text{eff}}}{(\omega-Mk)J_m(\alpha) - i\alpha J'_m(\alpha)Z_{\text{eff}}}\right)J_m(\alpha r_0)J_m(\alpha r)$$
$$P = \frac{(\omega-Mk_*)}{4}\left(\frac{q}{\omega-\omega_f} + \frac{q^*}{\omega+\omega_f^*}\right)$$

where $\mathcal{K}_\pm$ corresponds to the set of upstream/downstream poles as determined by the Briggs–Bers method. That is the set of poles which end up in the upper/lower half plane as $\Im(\omega_f) \to -\infty$. If $x > x_0$ we take the negative sign and $\mathcal{K}_-$ whereas if $x < x_0$ we take the positive sign and $\mathcal{K}_+$.

When we invert the $\omega$ Laplace transform we will get a contribution from the pole at $\omega_f$ and a contribution from any $\omega$ poles from the $k$-residue term. These additional poles however will correspond to transient modes, so if we are interested in the long time solution we expect to only have to consider the pole at $\omega_f$. We can then find $p_0(t,r,x,\theta)$ at long time:

$$p_0 = \sum_{\omega\in\mathcal{Q}} \sum_{m=-\infty}^{\infty} \mp \sum_{k_*\in\mathcal{K}_\pm} P e^{i(\omega t - k_*(x-x_0) - m\theta)} \left[ \frac{(\omega - Mk_*)Y_m(\alpha) - i\alpha Y_m'(\alpha)Z_{\text{eff}}}{\frac{\partial}{\partial k}((\omega - Mk)J_m(\alpha) - i\alpha J_m'(\alpha)Z_{\text{eff}})|_{k=k_*}} \right] J_m(\alpha r_0)J_m(\alpha r)$$
$$P = \frac{q(\omega - Mk_*)}{4}$$

where $\mathcal{Q}$ is the set of forcing frequencies $\{\omega_f\}$. Note that for each forcing frequency we have replaced the complex conjugate in $P$ by including $-\omega_f^*$ in $\mathcal{Q}$, which is equivalent due to the symmetry of the solution.

### D. Weak Nonlinearity

So far we have only looked at the linear acoustics that arise due to a point source. This linear solution is valid provided $\varepsilon/\delta_L \ll 1$ so that the nonlinear terms are much smaller than the linear acoustics. In aircraft engines the acoustics are often very loud which means that this assumption may not be true. Here we will consider a weakly nonlinear perturbation to find a solution for the nonlinear acoustics.

For the linear acoustics we solved a system of equations of the form:

$$L(p) = q(t)$$

where $L$ is a linear operator acting on $p$. If we consider the nonlinear terms we now have an equation of the form:

$$L(p) = Q(p,p) + q(t)$$

where $Q$ is quadratic in $p$. Now using the weakly nonlinear approximation we can decompose the problem into the linear acoustics problem above and a linear problem forced by nonlinear quantities of the linear solution:

$$L(p_0) = q(t) \qquad\qquad L(p_2) = Q(p_0, p_0)$$

Now we know the general form of $p_0$:

$$p_0 = \sum_{m=-\infty}^{\infty} \sum_{\omega\in\{\omega_f,\omega_*(m)\}} \sum_{k_*(\omega,m)\in\mathcal{K}_\pm} \mp Res_{\omega_*}(Res_{k_*}(\tilde{p}_0))e^{i(\omega t - kx - m\theta)}$$

where $\omega_*(m)$ are any transient modes and $k_*(\omega,m)$ are the spatial downstream/upstream modes. If we only consider the long time solution we can ignore the $\omega_*(m)$ terms. We can then consider quadratic quantities of $p_0$:

$$p_0 p_0 = \sum_{m_1,m_2=-\infty}^{\infty} \sum_{(\omega_*),(v_*)\in\mathcal{Q}} \sum_{k_*,l_*\in\mathcal{K}_\pm} Res_{\omega_*}(Res_{k_*}(\tilde{p}_0))Res_{v_*}(Res_{l_*}(\tilde{p}_0))e^{i(\Omega t - Kx - \mathcal{M}\theta)}$$

where $\Omega = \omega_* + v_*$, $K = k_* + l_*$, $\mathcal{M} = m_1 + m_2$ and each sum is now a double sum over every pair of frequencies/wavenumbers $(\omega_*, v_*)$, $(k_*, l_*)$. We can then write down the form for $p_2$:

$$p_2 = \sum_{m_1,m_2=-\infty}^{\infty} \sum_{\omega_*,v_*\in\mathcal{Q}} \sum_{k_*(\omega_*),l_*(v_*)\in\mathcal{K}_\pm} Res_\Omega(Res_K(\tilde{p}_2(\Omega,K,\mathcal{M})))e^{i(\Omega t - Kx - \mathcal{M}\theta)}$$

and we can solve for $Res_{\Omega_*}(Res_{K_*}(\tilde{p}_2(\Omega,K,\mathcal{M}))) = \tilde{p}_2(\Omega,K,\mathcal{M})$ separately for each mode.

# III. Matching

Now for each $\omega_* \neq v_*$ or $k_* \neq l_*$ there is a pair of pairs $(\omega_*, k_*; v_*, l_*)$ and $(v_*, l_*; \omega_*, k_*)$ in the sum that will give the same term, so we can combine these contributions when solving and include a factor of 1/2 to avoid double counting.

In region III, the equations for the leading order weakly nonlinear contribution are:

$$
\begin{aligned}
i(\Omega - MK)\tilde{\rho}_2 - iK\tilde{u}_2 + \frac{\tilde{v}_2}{r} + \tilde{v}_{2r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 &= iK\tilde{\rho}_0^*\tilde{u}_0 + \frac{i\mathcal{M}}{r}\tilde{\rho}_0^*\tilde{w}_0 - \frac{1}{r}(r\tilde{\rho}_0^*\tilde{v}_0)_r \\
i(\Omega - MK)\tilde{u}_2 - iK\tilde{p}_2 &= ik\tilde{u}_0^*\tilde{u}_0 + \frac{im}{r}\tilde{w}_0^*\tilde{u}_0 - \tilde{v}_0^*\tilde{u}_{0r} - i(\omega - Mk)\tilde{\rho}_0^*\tilde{u}_0 \\
i(\Omega - MK)\tilde{v}_2 + \tilde{p}_{2r} &= ik\tilde{u}_0^*\tilde{v}_0 + \frac{im}{r}\tilde{w}_0^*\tilde{v}_0 - \tilde{v}_0^*\tilde{v}_{0r} - i(\omega - Mk)\tilde{\rho}_0^*\tilde{v}_0 + \frac{1}{r}\tilde{w}_0^*\tilde{w}_0 \\
i(\Omega - MK)\tilde{w}_2 - \frac{i\mathcal{M}\tilde{p}_2}{r} &= ik\tilde{u}_0^*\tilde{w}_0 + \frac{im}{r}\tilde{w}_0^*\tilde{w}_0 - \tilde{v}_0^*\tilde{w}_{0r} - i(\omega - Mk)\tilde{\rho}_0^*\tilde{w}_0 - \frac{1}{r}\tilde{v}_0^*\tilde{w}_0 \\
i(\Omega - MK)\tilde{T}_2 - i(\Omega - MK)\tilde{p}_2 &= (ik\tilde{u}_0^* + \frac{im}{r}\tilde{w}_0^*)(\tilde{T}_0 - \tilde{p}_0) - \tilde{v}_0^*(\tilde{T}_0 - \tilde{p}_0)_r - i(\omega - Mk)\tilde{\rho}_0^*\tilde{T}_0
\end{aligned}
$$

where $\tilde{p}_0 = Res_{\omega_*}(Res_{k_*}(\tilde{p}_0(\omega, k))) = Res_{\omega_*}(Res_{k_*}(A(\omega, k)))J_m(\alpha r)$ and $\tilde{p}_0^* = Res_{v_*}(Res_{l_*}(\tilde{p}_0(\omega, k)))$ and similarly for all other leading order linear terms. All terms on the left hand side are of the same order and all terms on the right hand side are $\mathcal{O}(\varepsilon^2)$. When we combine pairs and substitute our solution for the linear problem this then becomes:

$$
\begin{aligned}
i(\Omega - MK)(\gamma\tilde{p}_2 - (\gamma - 1)\tilde{T}_2) - iK\tilde{u}_2 + \frac{\tilde{v}_2}{r} + \tilde{v}_{2r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 &= \frac{i(\Omega - MK)}{2}\tilde{p}_0^*\tilde{p}_0 - i(\Omega - MK)S \\
i(\Omega - MK)\tilde{u}_2 - iK\tilde{p}_2 &= iKS - \frac{iK}{2}\tilde{p}_0^*\tilde{p}_0 \\
i(\Omega - MK)\tilde{v}_2 + \tilde{p}_{2r} &= -S_r + \frac{1}{2}(\tilde{p}_0^*\tilde{p}_0)_r \\
i(\Omega - MK)\tilde{w}_2 - \frac{i\mathcal{M}\tilde{p}_2}{r} &= \frac{i\mathcal{M}}{2r}S - \frac{i\mathcal{M}}{2r}\tilde{p}_0^*\tilde{p}_0 \\
i(\Omega - MK)\tilde{T}_2 - i(\Omega - MK)\tilde{p}_2 &= -\frac{i(\Omega - MK)}{2}\tilde{p}_0^*\tilde{p}_0
\end{aligned}
$$

where $S = \dfrac{(kl + m^*m/r^2)\tilde{p}_0^*\tilde{p}_0 + \tilde{p}_{0r}\tilde{p}_{0r}^*}{2(\omega - Mk)(v - Ml)}$, which simplifies to:

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \aleph^2 - \frac{\mathcal{M}^2}{r^2}\right)\left(\tilde{p}_2 + S - \frac{1}{2}\tilde{p}_0^*\tilde{p}_0\right) = 0
$$

where $\aleph^2 = (\Omega - MK)^2 - K^2$. So in region III the leading order solution is:

$$
\tilde{p}_2 = DJ_{\mathcal{M}}(\aleph r) + \frac{1}{2}\tilde{p}_0^*\tilde{p}_0 - S \qquad\qquad \tilde{v}_2 = \frac{-\aleph DJ'_{\mathcal{M}}(\aleph r)}{i(\Omega - MK)}
$$

where $D$ is an arbitrary constant that will be found by matching between each layer and applying the impedance boundary condition at the wall of the duct.

Now in region II the leading order $\mathcal{O}(\varepsilon^2/\delta_L^2)$ equations are:

$$
\begin{aligned}
i(\Omega - UK)\tilde{T}_2 - \frac{T_\zeta\tilde{v}_2}{\delta_L} + \frac{T\tilde{v}_{2\zeta}}{\delta_L} + iKT\tilde{u}_2 &= \frac{i(\Omega - UK)}{T}\tilde{T}_0\tilde{T}_0^* + iK\tilde{T}_0^*\tilde{u}_0 + \frac{\tilde{v}_{0\zeta}\tilde{T}_0^*}{\delta_L} - \frac{\tilde{v}_0^*}{\delta_L}(\frac{2\tilde{T}_0T_\zeta}{T} - \tilde{T}_{0\zeta}) \\
i(\Omega - UK)\tilde{u}_2 - \frac{\tilde{v}_2U_\zeta}{\delta_L} &= ik\tilde{u}_0^*\tilde{u}_0 + \frac{\tilde{v}_0\tilde{u}_{0\zeta}^*}{\delta_L} + \frac{\tilde{T}_0^*}{T}(i(\omega - Uk)\tilde{u}_0 - \frac{\tilde{v}_0U_\zeta}{\delta_L}) \\
i(\Omega - UK)\tilde{T}_2 - \frac{\tilde{v}_2T_\zeta}{\delta_L} &= ik\tilde{u}_0^*\tilde{T}_0 + \frac{\tilde{v}_0\tilde{T}_{0\zeta}^*}{\delta_L} + \frac{\tilde{T}_0^*}{T}(i(\omega - Uk)\tilde{T}_0 - \frac{\tilde{v}_0T_\zeta}{\delta_L})
\end{aligned}
$$

When we substitute the results for the linear acoustics and combine pairs this becomes:

$$
\begin{aligned}
i(\Omega - UK)\tilde{T}_2 - \frac{T_\zeta \tilde{v}_2}{\delta_L} + \frac{T\tilde{v}_{2\zeta}}{\delta_L} + iKT\tilde{u}_2 &= \frac{C^*C}{2\delta_L^2}(iKT_\zeta U_\zeta + i(\Omega - UK)T_{\zeta\zeta}) \\
i(\Omega - UK)\tilde{u}_2 - \frac{\tilde{v}_2 U_\zeta}{\delta_L} &= \frac{C^*C}{2\delta_L^2}(iKU_\zeta^2 + i(\Omega - UK)U_{\zeta\zeta}) \\
i(\Omega - UK)\tilde{T}_2 - \frac{\tilde{v}_2 T_\zeta}{\delta_L} &= \frac{C^*C}{2\delta_L^2}(iKU_\zeta T_\zeta + i(\Omega - UK)T_{\zeta\zeta})
\end{aligned}
$$

which we can solve analytically to find at leading order:

$$
\tilde{v}_2 = -\frac{iKC^*C}{2\delta_L}U_\zeta - \frac{B}{\delta_L}i(\Omega - UK) \qquad \tilde{u}_2 = \frac{C^*C}{2\delta_L^2}U_{\zeta\zeta} - \frac{B}{\delta_L}U_\zeta \qquad \tilde{T}_2 = \frac{C^*C}{2\delta_L^2}T_{\zeta\zeta} - \frac{B}{\delta_L}T_\zeta
$$

In Region I we find that the forcing for the $\tilde{T}_2$ and $\tilde{v}_2$ equations are $\mathcal{O}(\varepsilon^2/\delta_L^2)$ while the forcing for the $\tilde{u}_2$ equation is $\mathcal{O}(\varepsilon^2/\sqrt{\varepsilon_k}\delta_L^2)$. This means that $\tilde{u}_2$ is independent of $\tilde{T}_2$ to $\mathcal{O}(\varepsilon_k)$ and $\tilde{v}_2$ is independent of $\tilde{T}_2$ to $\mathcal{O}(\sqrt{\varepsilon_k})$ so we have:

$$
iKT(0)\tilde{u}_2 + \sqrt{\frac{x}{\varepsilon_k x_0}}T(0)\frac{\tilde{v}_{2y}}{\delta_L} = \mathcal{O}(1)
$$

$$
i\left(\Omega - y\sqrt{\frac{\varepsilon_k x_0}{x}}U_\zeta(0)K\right)\tilde{u}_2 - \frac{\tilde{v}_2 U_\zeta(0)}{\delta_L} - k_r(\gamma-1)^2 T(0)^2\tilde{u}_{2yy} = \frac{\tilde{v}_0^*\tilde{u}_{0y}}{\delta_L}\sqrt{\frac{x}{\varepsilon_k x_0}} + ik\tilde{u}_0^*\tilde{u}_0 + \mathcal{O}(\sqrt{\varepsilon_k})
$$

where $\tilde{v}_2 \sim \mathcal{O}(\varepsilon^2/\delta_L)$ and $\tilde{u}_2 \sim \mathcal{O}(\varepsilon^2/\sqrt{\varepsilon_k}\delta_L^2)$ Now, assuming that the exponential term decays and $\Omega \neq 0$ we have at leading order:

$$
\begin{aligned}
\tilde{v}_2 &= -\frac{iB\Omega}{\delta_L} - \frac{iKC^*C}{2\delta_L}U_\zeta|_0\left(1 + \frac{\sqrt{\omega}+\sqrt{v}}{\sqrt{\Omega}}\mathrm{e}^{-\frac{y\sqrt{i\Omega}}{\sqrt{k_r(\gamma-1)T(0)}}} - \mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r(\gamma-1)T(0)}}} - \mathrm{e}^{-\frac{y\sqrt{iv}}{\sqrt{k_r(\gamma-1)T(0)}}}\right) \\
\tilde{u}_2 &= \frac{C^*CU_\zeta|_0\sqrt{ix}}{2\delta_L^2\sqrt{M}(\gamma-1)T(0)}\left(\sqrt{\omega}\mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r(\gamma-1)T(0)}}} + \sqrt{v}\mathrm{e}^{-\frac{y\sqrt{iv}}{\sqrt{k_r(\gamma-1)T(0)}}} - (\sqrt{\omega}+\sqrt{v})\mathrm{e}^{-\frac{y\sqrt{i\Omega}}{\sqrt{k_r(\gamma-1)T(0)}}}\right)
\end{aligned}
$$

we can then use the impedance boundary condition to solve for $D$ and we find that $\tilde{p}_2$ must be $\mathcal{O}(\varepsilon^2/\delta_L)$ to balance with $\tilde{v}_2$ which is amplified in the boundary layer to be $\mathcal{O}(\varepsilon^2/\delta_L)$ at the wall. This then gives:

$$
D\left(J_{\mathcal{M}}(\aleph) - \Omega Z(\Omega)\frac{i\aleph J'_{\mathcal{M}}(\aleph)}{(\Omega - MK)^2}\right) = \frac{C^*C}{2\delta_L}\left(iKZ(\Omega)U_\zeta|_0\left(1 - \frac{\sqrt{w}+\sqrt{v}}{\sqrt{\Omega}}\right)\right) \tag{4}
$$

So we find that the nonlinear pressure is a factor of $1/\delta_L$ greater than would be expected, due to the amplification of terms within the boundary layer. Now this solution has a singularity at $\Omega = 0$, which corresponds to the acoustic streaming modes. This means that we will have to include extra terms in the equations we solve to regularise near $\Omega = 0$.

## IV. Acoustic Streaming ($\Omega = 0$)

When $\Omega \to 0$, $Z(\Omega) \to \infty$ which means that when the boundary condition is applied at the wall we must have $\tilde{v}_2|_{y=0} = 0$. Our solution in region I for $\tilde{v}_2$ is singular and even if we rewrite it in terms of an integral of $\tilde{u}_2$, which now has a constant term due to the exponential of $\Omega$, we find that we still can't satisfy both the boundary condition at $y = 0$ and the matching to region II. Also we are ignoring a term of the form $\sqrt{\varepsilon_k}y\tilde{u}_2$ at leading order which will become large as $y \to \infty$ in this case. So we must now consider both leading order and next order $\mathcal{O}(\sqrt{\varepsilon_k})$ terms of the $\tilde{u}_2$ equation together. We can write the $\tilde{u}_2$ equation as:

$$
\begin{aligned}
&\left(\sqrt{\frac{x}{x_0\varepsilon_k}}\Omega/K - yU_\zeta(0)\right)\tilde{v}_{2y} + U_\zeta(0)\tilde{v}_2 - k_r(\gamma-1)^2T(0)^2\sqrt{\frac{x}{x_0\varepsilon_k}}\frac{1}{iK}\tilde{v}_{2yyy} \\
=&\frac{-C^*CU_\zeta|_0\sqrt{ix}}{2\delta_L\sqrt{M}(\gamma-1)T(0)}(-i\omega\sqrt{\omega}\mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r(\gamma-1)T(0)}}} + i\omega\sqrt{-\omega}\mathrm{e}^{-\frac{y\sqrt{-i\omega}}{\sqrt{k_r(\gamma-1)T(0)}}}) - \frac{iKC^*C}{2\delta_L}U_\zeta|_0^2 + \text{exponentials}
\end{aligned}
$$

which is correct to $\mathcal{O}(\sqrt{\varepsilon_k})$. Note that the first term on the right hand side is $\mathcal{O}(1/\sqrt{\varepsilon_k})$ while the exponentials that we have ignored are only $\mathcal{O}(1)$. We find that the dominant contribution to the particular solution comes from the first two terms. So we get:

$$\tilde{v}_2 = -\frac{iKC^*C}{2\delta_L}U_\zeta|_0\left(1 - \mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} - \mathrm{e}^{-\frac{y\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} + \tilde{v}_c\right) + \mathcal{O}(\sqrt{\varepsilon_k})$$

where $\tilde{v}_c$ is the solution to the homogeneous equation with $\tilde{v}_c(0) = 1$ and $\tilde{v}_c'(0) = -\frac{\sqrt{i\omega}+\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}$. This has solution:

$$\tilde{v}_c = Ay + By\int^y \frac{Ai'(x(-a)^{1/3})}{x^2}dx + Cy\int^y \frac{Bi'(x(-a)^{1/3})}{x^2}dx$$

where $a = \sqrt{\frac{\varepsilon_k x_0}{x}}\frac{iKU_\zeta|_0}{k_r(\gamma-1)^2T(0)^2}$. Now $Ai(x(-a)^{1/3}) \sim e^{-2/3x^{3/2}(-a)^{1/2}}$ and $Bi(x(-a)^{1/3}) \sim e^{2/3x^{3/2}(-a)^{1/2}}$ for $|\arg((-a)^{1/3})| < \pi/3$, so if we take the root of $-a$ that is in this arc then $Bi'(x)$ grows exponentially as $x$ increases. This means that to be able to match our solution to region II we must set $C = 0$.

Now $Ai'(x) = -\frac{1}{3^{1/3}\Gamma(1/3)} + \mathcal{O}(x^2)$ so we get:

$$\tilde{v}_c = 1 - \frac{\sqrt{i\omega}+\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}y - y\int_0^y \frac{-3^{1/3}\Gamma(1/3)Ai'(x(-a)^{1/3}) - 1}{x^2}dx$$

which then gives:

$$\tilde{v}_2 = -\frac{iKC^*C}{2\delta_L}U_\zeta|_0\left(2 - \mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} - \mathrm{e}^{-\frac{y\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} - \frac{y(\sqrt{i\omega}+\sqrt{-i\omega})}{\sqrt{k_r}(\gamma-1)T(0)} - y\int_0^y \frac{\frac{Ai'(x(-a)^{1/3})}{Ai'(0)} - 1}{x^2}dx\right)$$

Now to match to region II we consider the large $y$ behaviour of $\tilde{v}_2$. $\tilde{v}_2 \sim -\frac{iKC^*C}{2\delta_L}U_\zeta|_0(-y\frac{\sqrt{i\omega}+\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)} + \frac{\Gamma(1/3)(-a)^{1/3}}{3^{2/3}}y + 1)$, which must match to $-\frac{iKC^*C}{2\delta_L}U_\zeta + \frac{iBUK}{\delta_L}$ for small $\zeta$ in region II. This gives:

$$D = -\frac{(MK)^2}{\aleph J'_{\mathcal{M}}(\aleph)}\frac{C^*C}{2\delta_L}\frac{\sqrt{i\omega x}+\sqrt{-i\omega x}}{\sqrt{M}(\gamma-1)T(0)} + \mathcal{O}(\varepsilon_k^{-1/3}) \sim \mathcal{O}(\varepsilon_k^{-1/2})$$

(Note we could also include $\Omega$ in our solution by substituting $z = y - \Omega/KU_\zeta\sqrt{x/x_0\varepsilon_k}$ and we find that this regularised solution is valid for $\Omega \ll \varepsilon_k^{1/3}$ so $D$ will be at most $\mathcal{O}(1/\sqrt{\varepsilon_k})$.)

### A. Oscillating Streaming Solutions

So far for the acoustic streaming modes we have assumed that $|arg(-a)^{1/3}| < \pi/3$ so that the exponential terms decay. We will now consider the case where $|arg(-a)^{1/3}| = \pi/3$ (i.e. $|arg(-iK)| = \pi$) and these exponential terms now oscillate and must be considered when matching.

In region I the Airy function of the second kind will no longer decay, so the solution can now be written in the form:

$$\begin{aligned}
\tilde{v}_c &= 1 - \frac{\sqrt{i\omega}+\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}y - By\int_0^y \frac{\frac{Ai'(x(-a)^{1/3})}{Ai'(0)} - 1}{x^2}dx - (1-B)y\int_0^y \frac{\frac{Bi'(x(-a)^{1/3})}{Bi'(0)} - 1}{x^2}dx \\
\tilde{u}_2 &= -\frac{C^*CU_\zeta|_0\sqrt{x}}{2\delta_L^2\sqrt{M}(\gamma-1)T(0)}\left(\sqrt{i\omega}(\mathrm{e}^{-\frac{y\sqrt{i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} - 1) + \sqrt{-i\omega}(\mathrm{e}^{-\frac{y\sqrt{-i\omega}}{\sqrt{k_r}(\gamma-1)T(0)}} - 1)\right) \\
&+ \frac{C^*CU_\zeta|_0\sqrt{x}}{2\delta_L^2\sqrt{\varepsilon_k x_0}}\frac{(-a)^{2/3}}{Ai'(0)}\left(B\int_0^y Ai(x(-a)^{1/3})\mathrm{d}x + (1-B)\frac{Ai'(0)}{Bi'(0)}\int_0^y Bi(x(-a)^{1/3})\mathrm{d}x\right)
\end{aligned}$$

we can then use the following approximations for large $y$[15]:

$$Ai(y\mathrm{e}^{\pm\frac{i\pi}{3}}) \sim \frac{\mathrm{e}^{\mp\frac{2i}{3}y^{3/2}}}{2\sqrt{\pi}y^{1/4}\mathrm{e}^{\pm\frac{i\pi}{12}}}, \qquad Bi(y\mathrm{e}^{\pm\frac{i\pi}{3}}) \sim \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{\pm\frac{\pi i}{6}}}{y^{1/4}}\sin(\frac{2x^{3/2}}{3}+\frac{\pi}{4}\mp\frac{i}{2}\ln 2),$$

$$\implies \int_0^y Ai(x(-a)^{1/3})\mathrm{d}x \sim \frac{1}{3(-a)^{1/3}} - \frac{\mathrm{e}^{-\frac{2}{3}y^{3/2}\sqrt{-a}}}{2\sqrt{\pi}y^{3/4}(-a)^{7/12}},$$

$$\int_0^y Bi(x(-a)^{1/3})\mathrm{d}x \sim \frac{\mathrm{e}^{\frac{2}{3}y^{3/2}\sqrt{-a}}}{\sqrt{\pi}y^{3/4}(-a)^{7/12}} \pm i\left(\frac{1}{(-a)^{1/3}} - \frac{\mathrm{e}^{-\frac{2}{3}y^{3/2}\sqrt{-a}}}{2\sqrt{\pi}y^{3/4}(-a)^{7/12}}\right)$$

where the $\pm$ in the second integral corresponds to the sign of $arg((-a)^{1/3}) = \pm i\pi/3$. We can then use this to find the asymptotic behaviour of $\tilde{u}_2$ in region I for large $y$:

$$\tilde{u}_2 \sim \frac{C^* C U_\zeta|_0\sqrt{x}}{2\delta_L^2\sqrt{\varepsilon_k x_0}}\frac{(-a)^{1/12}}{\sqrt{\pi}y^{3/4}}\left(\frac{(1-B)}{Bi'(0)}\mathrm{e}^{\frac{2\sqrt{-a}}{3}y^{3/2}} - \mathrm{e}^{-\frac{2\sqrt{-a}}{3}y^{3/2}}\left(\frac{B}{2Ai'(0)} \pm i\frac{(1-B)}{2Bi'(0)}\right)\right) + \text{const} + \mathcal{O}(1/y^{5/4})$$

If we now consider the equations in region II for gradients $\sim 1/\sqrt{\varepsilon_k}$ to order $\mathcal{O}(\sqrt{\varepsilon_k})$, ignoring $\tilde{T}$ terms which oscillate with a different frequency and will contribute to an additional solution for $\tilde{v}$ but not for $\tilde{u}$, we have:

$$\frac{\tilde{v}_{2\zeta}}{\delta_L} + iK\tilde{u}_2 = \mathcal{O}(\sqrt{\varepsilon_k})$$

$$-iKU\tilde{u}_2 - \frac{\tilde{v}_2 U_\zeta}{\delta_L} - \left(\frac{M(\gamma-1)^2T^2}{x}(\tilde{u}_{2\zeta\zeta} + \frac{T_\zeta}{T}\tilde{u}_{2\zeta}) + \tilde{u}_{2\zeta}(\frac{V}{\delta_L} + \frac{U\zeta}{2x})\right) = \mathcal{O}(\varepsilon_k)$$

We can try a multiple scales solution of the form:

$$\tilde{v}_2 = \frac{F_\pm(\zeta)}{\delta_L}\mathrm{e}^{\pm\int^\zeta\sqrt{\frac{-iKUx}{M(\gamma-1)^2T^2}}\mathrm{d}\zeta} \qquad\qquad \tilde{u}_2 = \frac{G_\pm(\zeta)}{\delta_L^2}\mathrm{e}^{\pm\int^\zeta\sqrt{\frac{-iKUx}{M(\gamma-1)^2T^2}}\mathrm{d}\zeta}$$

and we find:

$$F_\pm(\zeta) = \pm\frac{\sqrt{-iKM(\gamma-1)^2T^2}G_\pm(\zeta)}{\sqrt{Ux}}$$

$$G_\pm(\zeta) = \frac{G_\pm}{U^{3/4}}\mathrm{e}^{-\int_0^\zeta\frac{x}{2M(\gamma-1)^2T^2}(\frac{U\zeta}{2x}+\frac{V}{\delta_L})\mathrm{d}\zeta} = \frac{G_\pm}{U^{3/4}}\mathrm{e}^{-\int_0^{\eta(\zeta)}\frac{f}{4}\mathrm{d}\eta}$$

where we are assuming here that $U$, $V$ and $T$ are given by the compressible Blasius boundary layer and thus $f(\eta)$ satisfies the Blasius equation:

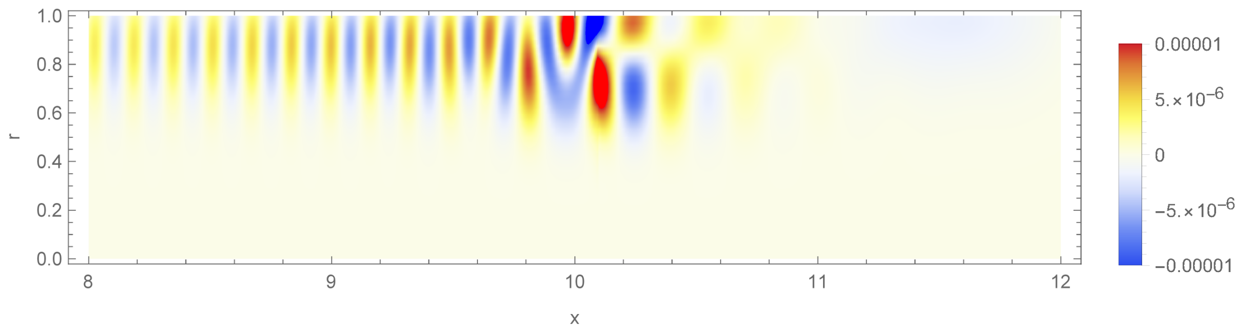
$$U = Mf', \quad V = -\frac{M\delta_L}{2x}(\zeta f' - f/\eta'), \quad \eta' = \rho = 1/(\gamma-1)T, \quad f''' + \frac{ff''}{2} = 0, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty$$

This means that our additional large oscillatory term in $\tilde{u}_2$ decays exponentially due to the integral term in $G$ which arises only because we are including the non-parallel contributions. This means that the large oscillatory streaming solution is confined to within the boundary layer. This shows that it is necessary to consider the non-parallel nature of the boundary layer when solving for the nonlinear modes and the previously found large oscillatory behaviour that extended to the centre of the duct was a artefact of the parallel flow assumption.

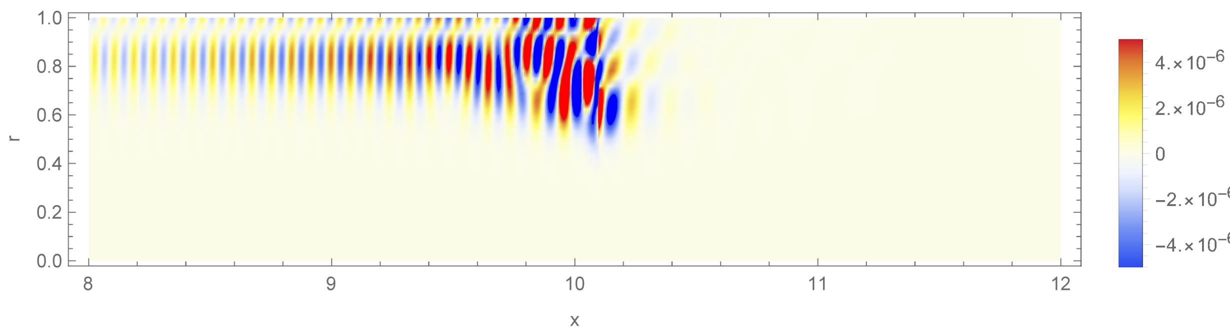
## V. Results

To calculate the acoustic pressure field we truncate the sum over all streamwise $k$-modes for $|Im(k_*)| > N$ and we will only plot the contribution from a single azimuthal ($m$) mode.

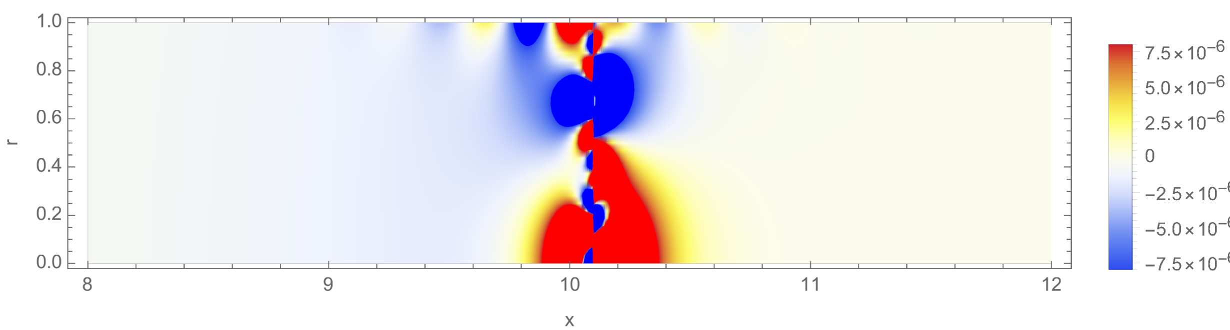
Figure 3 shows the separate components of the acoustic pressure field for a source at $(r_0, \theta_0, x_0) = (0.8, 0, 10.1)$ with $\omega = 10$, $m = 10$ and $Z = 3 + 1.5i\omega - 1.15i/\omega$ we use $\gamma = 1.4$, $\delta = 10^{-3}$ and $\varepsilon = 10^{-5}$. It can be observed that there is some distortion at $x = x_0$ due to the truncation of the sum.




(a) Linear pressure $p_0$




(b) Nonlinear acoustic pressure $p_2$




(c) Nonlinear streaming pressure $p_2$

Figure 3: Linear and Nonlinear Acoustics at $t = 0.5$ for a source at $(r_0, \theta_0, x_0) = (0.8, 0, 10.1)$ with $\omega = 10$, $m = 10$ and $Z = 3 + 1.5i\omega - 1.15i/\omega$, taking only the first four $k$ modes in each direction

## VI. Conclusion

We have shown that the nonlinear modes that arise due to the linear acoustics are a factor of $1/\delta_L$ greater than would be expected for the nonlinear modes in a hard walled duct. This is a consequence of the amplification and subsequent interaction of certain quantities in the boundary layer. We have shown that the previously found large $\mathcal{O}(\varepsilon^2/\delta_L)$ oscillatory nonlinear solution was an artefact of the parallel flow assumption. However we have also shown that even when taking into account the non-parallel effects an $\mathcal{O}(\varepsilon^2/\delta_L)$ amplified nonlinear solution is still permitted. We have also shown that the nonlinear streaming modes are amplified further by an additional factor of $1/\sqrt{\varepsilon_k}$.

Our results also show that for the nonlinear acoustics the outer pressure does not obey the effective impedance boundary condition due to the amplifying mechanisms within the boundary layer. This means that a single effective impedance boundary condition will not adequately resolve both the linear and the nonlinear acoustics as they have different effective impedances.

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