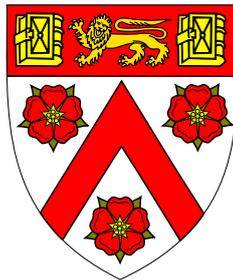# Linear and weakly nonlinear boundary layer acoustics in a lined duct

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This dissertation is submitted for the degree of

Doctor of Philosophy

Trinity College

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To Mum and Dad

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Owen David Petrie
April 2019

Acknowledgements

Linear and weakly nonlinear boundary layer acoustics in a lined duct

Owen David Petrie

In this thesis I look at the effect of boundary layer flow on the acoustics of an acoustic lining. Acoustic linings are used in aircraft engine ducts to reduce the sound they produce. They typically consist of an array of Helmholtz resonators that are characterised by their impedance - a linear relationship between the acoustic pressure and acoustic normal velocity. However in aircraft engines the air in the duct is moving quickly over the lining and so there is a boundary layer near the lining. The impedance boundary condition then needs to be modified to take into account the effect of the boundary layer flow on the acoustics.

In this thesis I begin by considering the weakly nonlinear acoustics for a parallel visco-thermal boundary layer flow with uniform geometry over an acoustic lining. This is done using a two layer matched asymptotics model that is solved numerically. It is known that certain linear acoustic components are amplified within the boundary layer and I show that this causes the weakly nonlinear acoustics to be amplified outside of the boundary layer. I also show that this model leads to some surprising large rapidly oscillating disturbances that propagate out into the centre of the duct in certain cases.

I then consider the case of a non-parallel boundary layer with non-uniform geometry. This is done using a three-layer WKB asymptotic solution and the corresponding boundary condition is derived and shown be in agreement with previous work in certain limits. I also then show that for the non-parallel case the weakly nonlinear acoustics, while still amplified, do not display the large oscillating behaviour, suggesting that it is important that non-parallel effects are considered.

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Nomenclature

Roman Symbols

c_p specific heat at constant pressure

c_V specific heat at constant volume

$\tilde{p}$ acoustic pressure perturbation

$\tilde{T}$ acoustic temperature perturbation

$\tilde{u}$ acoustic streamwise velocity perturbation

$\tilde{v}$ acoustic radial velocity perturbation

$\tilde{w}$ acoustic azimuthal velocity perturbation

p boundary layer mean flow pressure

T boundary layer mean flow temperature

u boundary layer mean flow streamwise velocity

v boundary layer mean flow radial velocity

c_s outer mean flow speed of sound

M outer mean flow Mach number

P outer mean flow pressure

$\bar{T}$ outer mean flow temperature

U outer mean flow streamwise velocity

V outer mean flow radial velocity

q mass source strength

$\mathcal{K}_{\pm}$ set of upstream/downstream streamwise modes

$\mathcal{W}$ set of forcing frequencies

x streamwise coordinate

Pr Prandtl number

Re Reynolds number

Greek Symbols

$\tilde{\rho}$ acoustic density perturbation

γ ratio of specific heats

ρ mean flow density

Superscripts

$*$ dimensional variable

$\star$ complex conjugate

Chapter 1

Introduction

1.1 Motivation

The acoustics in flow over acoustic linings is of interest due to their use in aeroengine ducts. Acoustic liners are an essential part of civilian aircraft engines as aircraft engine manufacturers rely on acoustic linings in the inlets and outlets of their engines to reduce the noise [23]. Thus a good understanding of their effect is necessary to get the best possible reduction in noise.

As shown in figure 1.1, while aircraft have become quieter over the last few decades the noise limit regulations have become ever stricter and are set to become increasingly strict in the coming years. As a consequence much work is needed to meet these requirements. A better understanding of acoustic liners is needed to enable the aircraft manufacturers to meet these newer noise requirements.

Noise is an important concern for the aviation industry [13], particularly for commercial airliners which operate from busy airports with ever more stringent noise control policies. The major contributor to an aircraft's noise is its engines. Historically low bypass ratio engines were used and the majority of the noise produced was from the jet noise. However more recent engine developments have allowed the construction of larger high bypass ratio engines which are significantly more fuel efficient. A result of this however is that the major contributor to the noise of the engine is now the inlet and outlet of the engine duct rather than the jet noise. This is because high bypass ratio engines push a larger mass of air at a slower velocity to achieve the same momentum flux, which reduces the effect of jet noise [21].

Figure 1.2 shows the relative amplitude of the main contributors to aircraft engine noise in a 1992-technology level airliner. It can be seen that the largest source of noise is the engine inlet and outlet.

Figure 9.1: Historic and future trends in cumulative certificated aircraft noise levels from noise discussion paper, 1960-2040
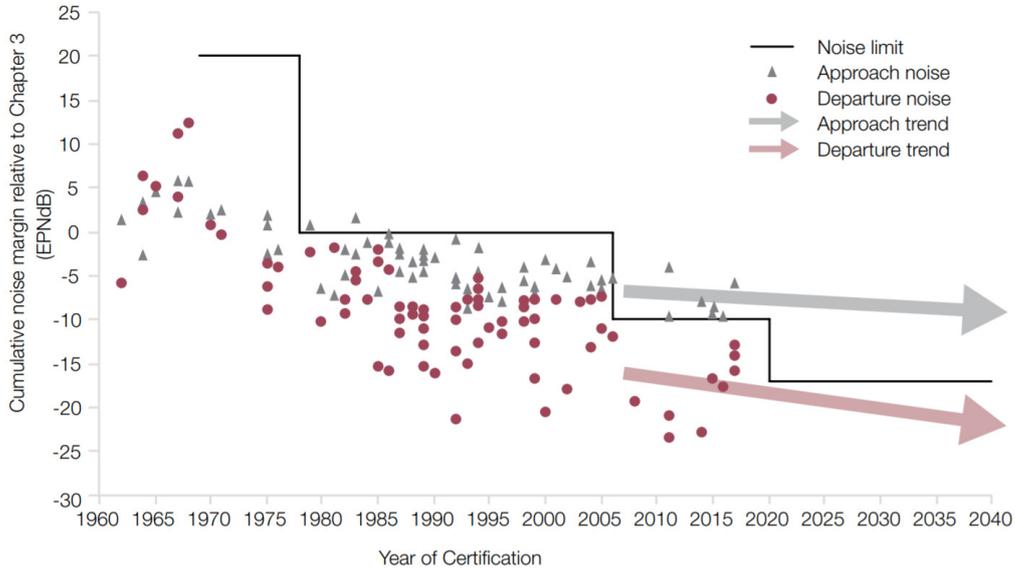
Fig. 1.1 Historic aircraft noise reduction and noise regulation reduction [8, pg. 169]
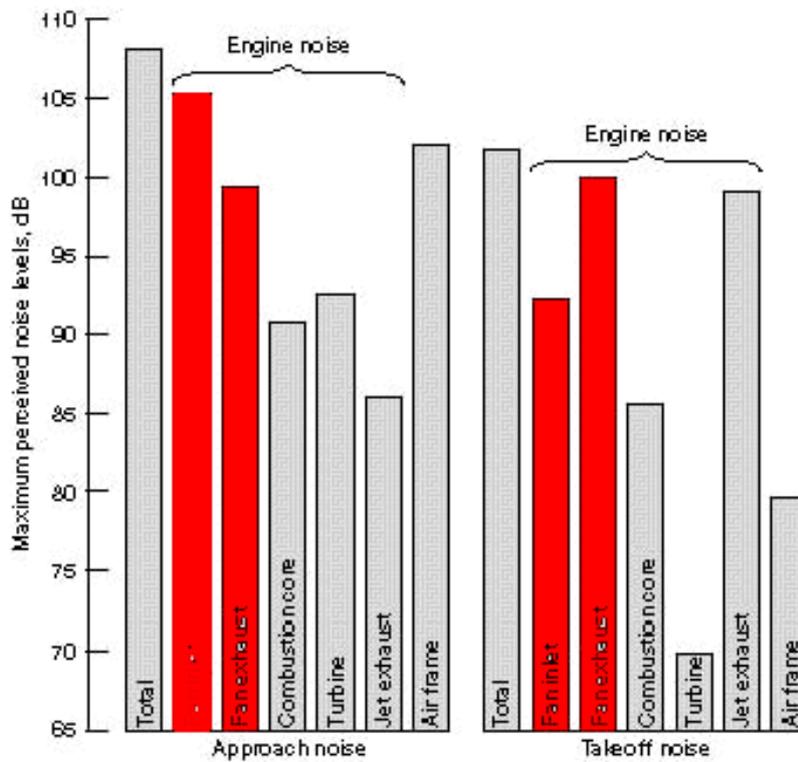
Fig. 1.2 Breakdown of noise components for a 1992-level technology engine at takeoff and approach. Credit: NASA [12]

Newer aircraft use acoustic linings in the engine duct to try and reduce the noise that is produced. These have proved to be very effective. However as fuel efficiency is ever more important, newer engines have had ever larger diameters to increase fuel efficiency and ever shorter lengths of ducting to reduce drag. This in turn reduces the effective area of the acoustic linings. It is therefore desirable to have an effective way of modelling the acoustic linings so that the best possible noise reduction may be obtained over the shortest possible distance.

1.2   Acoustic linings

So far I have outlined the reason acoustic linings are used but not described what they are or how they work. In this section I will describe the typical construction of an acoustic lining and describe how they may be modelled.

Typically the acoustic lining in an aircraft engine consists of a perforated metal panel beneath which is a honeycomb array of acoustic resonators which are capped at the end. A cutaway diagram can be seen in figure 1.3. The acoustic lining is characterised by its impedance $Z(\omega)$, which is a linear relationship between the acoustic pressure perturbation $\tilde{p}$ and acoustic normal velocity $\tilde{v}$ at the lining

$$\tilde{p} = Z(\omega)\tilde{v}. \tag{1.1}$$

Note that Z typically depends on the frequency ω of the acoustics and so this boundary condition can only be applied in the frequency domain.

To derive the form for the impedance of the liner we need to consider the mechanics of the individual cells of the liner. Each of the cells acts as a Helmholtz resonator. A Helmholtz resonator consists of a cavity filled with air which has a small opening, called a neck.

One of the simplest physically-realizable ways of modelling the impedance of a Helmholtz resonator is by using a mass-spring-damper model. This is because the action of the resonator can be described as the oscillation of the mass of fluid in the neck against the 'spring' of compressible fluid inside the cavity with vortex shedding and viscous dissipation in the neck causing damping. Figure 1.4 shows a diagram of a single Helmholtz resonator showing the vortex shedding off the edge of the opening which is one of the causes of the damping.

Typically the damping is fairly low and a nearly resonant mode for the resonator exists. For this nearly resonant mode the dissipative effects will be large, nonlinear effects of the liner will be important and the sound power will be greatly reduced. It is often desirable to tune the resonator such that the nearly resonant mode occurs at the same frequency as
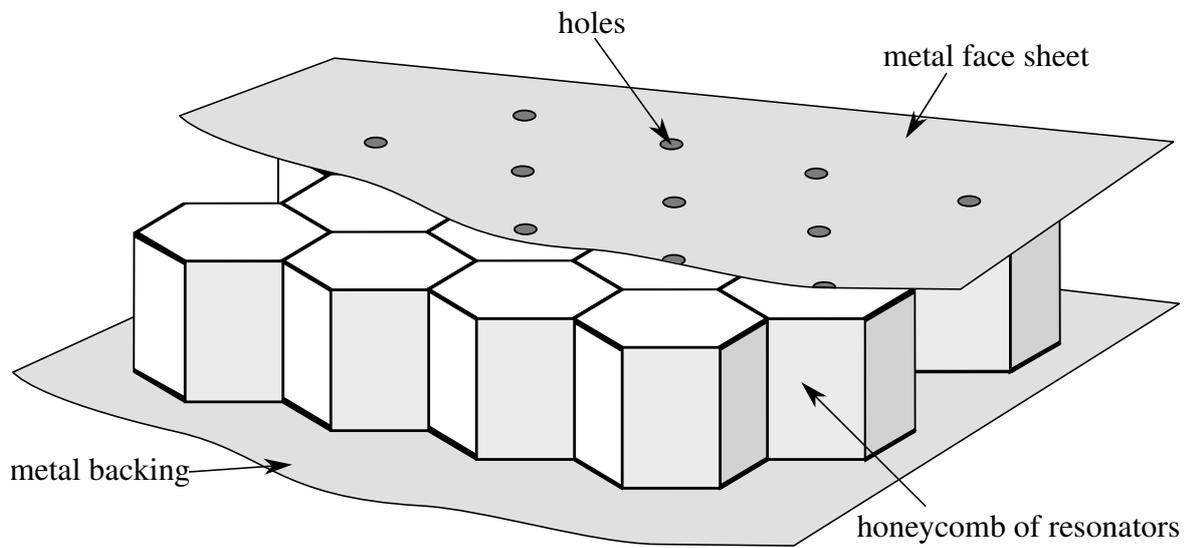
Fig. 1.3 Cutaway diagram of an acoustic lining
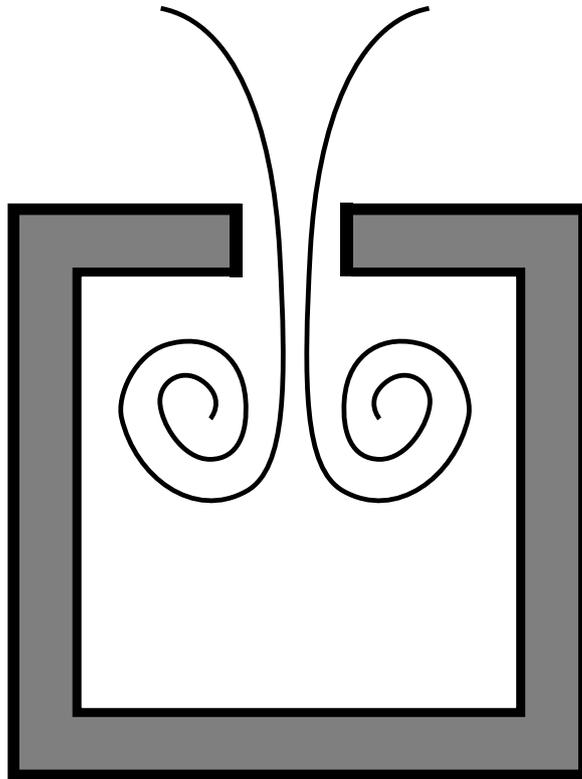
Fig. 1.4 Diagram of a single Helmholtz resonator

the largest source of noise inside the engine. However this only produces a narrowband sound absorption for frequencies near the resonant frequency. Independently of the linear or nonlinear reaction of the liner, the boundary layer above the liner can have important nonlinearities at all frequencies, and this is what I am interested in investigating in this thesis.

It has been shown [32] that, to good approximation, the liner may be modelled by a linear boundary condition. For this boundary condition it is assumed that the effect of each individual resonator is averaged across the liner which leads to the following boundary condition in the time domain for the acoustic lining

$$\frac{\partial p}{\partial t} = d\frac{\partial^2 v}{\partial t^2} + R\frac{\partial v}{\partial t} + bv, \tag{1.2}$$

where d corresponds to the mass of the air in the neck, b the compressibility of the fluid in the resonator and R the damping. Here the normal velocity acts as a damped oscillator which is forced by the changes in pressure. As this boundary condition involves derivatives of flow quantities and is therefore difficult to solve we typically map to the frequency domain, using the Fourier conventions outlined in section §2.5. In the frequency domain the boundary condition can be written as an algebraic equation in terms of the impedance, as in equation (1.1), with

$$Z(\omega) = R + i\omega d - \frac{ib}{\omega}, \tag{1.3}$$

where $\mathrm{Re}(Z) = R$ is known as the resistance and $\mathrm{Im}(Z) = \omega d - b/\omega$ is known as the reactance. By solving for the eigenfunctions of the impedance condition we find the closest real frequency to the impedance resonant frequency to be $\sqrt{\frac{b}{d}}$ and the impedance damping factor to be $\frac{R}{4bd}$ [6].

The impedance of an acoustic lining is typically measured using a normal impedance tube, e.g. [2]. This uses a sound source perpendicular to the lining to produce different frequencies for which the reaction of the lining can be measured. There also exist numerical methods to evaluate the impedance [33]. However in a normal impedance tube there is no background flow and so we need a way to convert the impedance measured or numerically simulated in the normal impedance tube to an effective impedance for linings in aircraft engines with flow passing over them.

Given a liner that has a known impedance when there is no-flow, we wish to find a way to derive its effective impedance when there is a grazing flow over the lining. The effective impedance is defined by the impedance the outer flow, the flow in the duct outside the boundary layer, 'sees' at the wall when there is a boundary layer flow at the wall

$$\tilde{p}_O(r = a) = Z_{\mathrm{eff}}\tilde{v}_O(r = a), \tag{1.4}$$

where the wall of the duct is at $r = a$ and $\tilde{p}_O$ and $\tilde{v}_O$ are the outer acoustics from the centre of the duct extrapolated through the boundary layer. All the boundary layer effects are then contained within Z_{eff}. Note that Z_{eff} may now also depend on the wavenumber of the acoustics and on the background boundary layer flow profile.

The effective impedance of a lining may be measured in a grazing flow impedance tube (GFIT) [14] however this is both time consuming and expensive. Also the scale and aspect ratio of the GFIT apparatus is very different to an actual aircraft engine and so it is difficult to measure the effective impedance in the relevant physical regimes.

Much work has been carried out trying to model the effective impedance by matching through the boundary layer and using the known no-flow impedance results. This is because given the effective impedance only the acoustics in the uniform flow would need to be considered and the boundary layer would not need to be resolved to accurately predict the acoustics. This would make the simulation of the acoustics in a duct much simpler to carry out accurately. Some of these boundary condition models are covered in the next section.

1.3   Previous grazing flow impedance boundary conditions

Historically the Myers boundary condition [24] has been used to model the effective boundary condition for flow over acoustic linings. It assumes that both the acoustic pressure and acoustic normal displacement are constant across the boundary layer. This then gives the Myers effective impedance

$$Z_{\text{eff}}^{\text{Myers}} = \frac{Z\omega}{\omega - Uk}, \tag{1.5}$$

where Z is the no-flow impedance of the lining, U is the outer uniform flow velocity and k is the streamwise wavenumber of the acoustics. However the Myers boundary neglects the effects of shear, viscosity and non-parallel effects inside the boundary layer as it assumes an infinitely thin boundary layer at the wall. This means that the Myers boundary condition gives a vortex sheet at the boundary. Furthermore it has been shown to be ill-posed [3, 11]. This is because it admits surface modes for which there is no maximum exponential temporal growth rate as $k \to \infty$, i.e. there can be arbitrarily large growth in time for arbitrarily short wavelengths. This means that better models of the acoustics in flows over acoustic linings are needed.

More recent work [5] gave a modified Myers boundary condition which took account of the shear in the boundary layer of the background flow but still ignored the effect of viscosity.

This gave a closed form solution

$$Z_{\text{eff}}^{\text{Brambley}} = \frac{Z\omega}{\omega - Uk} \frac{1 - \frac{i(\omega - Uk)^2}{Z\omega} \delta I_0^B}{1 + \frac{i\omega Z(k^2 + m^2)}{(\omega - Uk)^2} \delta I_1^B}, \tag{1.6}$$

involving the integrals δI_0 and δI_1 over the mean flow boundary layer profile

$$\delta I_0^B = \int_0^1 1 - \frac{(\omega - uk)^2}{(\omega - Uk)^2} \, dr,$$

$$\delta I_0^B = \int_0^1 1 - \frac{(\omega - Uk)^2}{(\omega - uk)^2} \, dr,$$

where u is the radially varying boundary layer mean flow velocity, U is the outer uniform mean flow velocity and the radius of the duct has been nondimensionalised to be at $r = 1$ here.

However it has been shown [16] that the effect of viscosity on the acoustics occurs at the same order of magnitude as shear, and so viscosity must also be taken into account. This requires considering a finite thickness viscous boundary layer near the acoustic lining and solving for the acoustics in the boundary layer and then matching to an outer solution, which is an inviscid solution uniform flow [18, 17, 19]. This approach is used in the first part of chapter §3 to derive the leading order linear acoustics in a parallel duct. It has been shown that this approach agrees more closely to results obtained by solving the linearised Navier Stokes equations for the entire duct.

All of the above methods only consider the linear acoustics in a lined duct. However as discussed in section §1.1 aircraft engines are very loud and the assumption of linearity may not be valid. However, even if the sound within the engine ducting may be considered linear, an amplification mechanism by a factor of $1/\delta$, where δ is the boundary layer thickness (typically $\delta = 10^{-3}$), has been shown to exist within a thin visco-thermal boundary layer [4]. Experimental evidence also suggests nonlinearity becomes important at lower amplitudes than expected for flow over an acoustic lining [1]. This amplification is due to the interaction of the acoustic normal velocity with the large shear within the boundary layer.

A lot of previous work has been carried out to investigate the stability of perturbations in flow over hard walls. For the case of incompressible perturbations in flows over hard walls it has been shown that that non-parallel effects are important [27, 10] and some investigations into nonlinear effects have been made [36]. For the case of compressible perturbations there has also been a lot of prior work, e.g. [28]. In this case it has also been shown that non-parallel

effects are important to consider [35] and again some investigations into nonlinear effects have been made [22].

However it should be noted that the amplification effect does not occur for a hard walled duct as the no-penetration condition at the wall in this case causes the acoustic normal velocity to be small near the wall and thus suppresses the amplification mechanism. This means that while a lot of work has been carried out on the stability of perturbations in flows over hard walls, these cases will not display the amplification within the boundary layer and so constitute a different physical regime.

There has been some work on compressible perturbations in boundary layer flows with non-hard walls [37, 34]. However in this previous work the normal velocity at the wall was prescribed and so this doesn't correspond to the case of a locally reacting acoustic liner. This is because the prescribed normal velocity at the wall drives the perturbations rather than the perturbations being driven by an outside pressure source that interacts with the wall through an impedance condition and so this is again in a different physical regime. It is clear then that for the case of acoustic linings there is still scope to investigate further.

The amplification mechanism discussed above motivates the investigation into the weakly nonlinear acoustics for a lined duct with parallel mean flow in chapter §3. In that chapter I will show that the nonlinearity can cause unexpected acoustic streaming phenomena in certain cases. Acoustic streaming is a well studied phenomena in the case of no flow and hard walls [29], but there has been very little previous work done on acoustic streaming in flow over acoustic liners.

All of the previous models for acoustic linings discussed above assume the flow through the duct is parallel. This assumption is valid for large distances downstream and so may work well for experiments conducted on high aspect ratio apparatus (e.g. the grazing flow impedance tube). However it is less likely to be valid for aeroengines whose aspect ratio is moderate (particularly since the diameter of turbofan engines has historically increased). Furthermore it has been suggested [10] that the non parallel effects of developing boundary layers may have an important effect on the acoustics. However this previous work considered only no-penetration boundary conditions for the acoustics which as I have discussed are in a different scaling regime to acoustics in boundary layers near impedance linings. This motivatives the investigation in chapters §4 and §5, into the linear and weakly nonlinear acoustics respectively for a developing non-parallel boundary layer profile, to see whether the acoustic streaming phenomena for the parallel flow case are an artefact of the parallel assumption or an actual physical effect.

1.4   Summary

In this thesis I will investigate the weakly nonlinear acoustics of boundary layer flow over acoustic linings. I will show that the current assumption of parallel flow used to solve for the linear acoustics results in some unusual non-physical nonlinear acoustics. I will then develop a three-layer WKB method to solve for both the linear and weakly nonlinear acoustics in the duct with a slowly developing boundary layer and show that this method does not give these unphysical results.

In chapter §2 I will introduce the governing equations for the acoustics in a duct and then solve for the outer uniform mean flow for the cases of a parallel and a non-parallel duct. I will then derive the compressible Blasius boundary layer solution for the inner mean flow in both cases.

In chapter §3, I first cover the derivation of the leading order linear acoustics for the case of a lined duct with parallel mean flow. I will then extend this numerical solution to solve for the weakly nonlinear acoustics that arise. I will show that both the double frequency harmonic and zero frequency streaming modes are amplified due to the amplification mechanism within the boundary layer. I will then also show that in certain cases the acoustic streaming modes can have very large, highly oscillatory solutions that propagate out into the centre of the duct.

As it it believed that the highly oscillatory streaming solutions are an artefact of the parallel mean flow assumption, in chapter §4 I will develop a global, slowly varying WKB solution for the linear acoustics within a lined duct of varying radius with a developing boundary layer mean flow. I will consider the acoustics sufficiently far downstream so that $1/k \ll x$, where k is the streamwise wavenumber, which means that the boundary layer is sufficiently well developed that the effect of viscosity on the acoustics is restricted to an inner-inner region. This means that a three-layer formulation may be used that can be solved analytically using asymptotic matching.

In chapter §5 I will then extend the slowly varying solution further to investigate the weakly nonlinear non-parallel acoustics. I again find that the weakly nonlinear acoustics are amplified but now the highly oscillatory acoustic streaming solution is confined to the inner boundary layer regions.

I will then conclude and discuss potential further work that could be undertaken to extend this work.

Chapter 2

Formulation

To investigate the acoustic properties of boundary layer flow over acoustic linings I will be considering the problem of acoustics in a duct of circular cross section, lined with an acoustic lining. In chapter §3 I will restrict to the case of a parallel cylindrical duct with the assumption of a sufficiently well developed parallel boundary layer flow. As discussed in the introduction, §1.1, the linear acoustics in this case have been studied extensively. Here I will extend this analysis to the weakly nonlinear acoustics that arise. I will show that under these assumptions a surprising large rapidly varying streaming mode may propagate into the centre of the duct in certain cases. To investigate whether this effect is a physical phenomenon or just a consequence of the assumptions in chapters §4 and §5 I will develop the solution for acoustics in a non-parallel boundary layer flow for the linear and weakly nonlinear acoustics respectively. I also allow the radius of the duct to vary slowly in these chapters and derive an analytic solution that considers the effects of shear, viscosity and streamwise variations.

In this chapter I will introduce the governing equations for the above problems. I will derive the solution for the steady outer mean flow in a duct of slowly varying radius. I will then derive the solution for the steady viscous boundary layer flow near the wall of the slowly varying radius duct. These mean flow solutions are necessary so that a perturbation expansion for the small amplitude unsteady acoustic quantities can be used to solve for the acoustics in the following chapters.

2.1   Nondimensionalisation

To begin it is useful to nondimensionalise all the physical parameters inside the duct. I nondimensionalise lengths with respect to the radius of the duct l^*, speeds with respect to the mean flow sound speed c_0^*, density with respect to the mean flow density ρ_0^* and specific heats with respect to the specific heat at constant pressure c_p^*. The values l^*, c_0^*, ρ_0^* and c_p^*

Density	$\rho^* = \rho_0^* \rho$	Pressure	$p^* = c_0^{*2} \rho_0^* p$
Velocity	$\mathbf{u}^* = c_0^* \mathbf{u}$	Viscosity	$\mu = c_0^* l^* \rho_0^* \mu$
Distance	$x^* = l^* x$	Thermal Conductivity	$\kappa^* = c_0^* l^* \rho_0^* c_p^* \kappa$
Time	$t^* = l^*/c_0^* t$	Temperature	$T^* = c_0^{*2}/c_p^* T$

Table 2.1 Dimensional and nondimensional variables, where $*$ denotes a dimensional variable, with lengthscale l^*, velocity c_0^*, density ρ_0^* and specific heat at constant pressure c_p^*.

are taken along the centreline of the duct and for the case of a slowly varying duct, are taken at the origin $x = 0$. This means that for the parallel duct the mean flow density will be $\rho = 1$ and the mean flow velocity $U = M$ is the Mach number of the flow. While for the slowly varying duct this is only true at $x = 0$. The nondimensionalisation of all physical parameters is shown in table 2.1.

In the duct we will use cylindrical coordinates (r^*, θ, x^*) which after the nondimensionalisation become (r, θ, x). r is the radial coordinate, θ the azimuthal coordinate and x is the axial coordinate. The wall of the duct is at $r = a(x)$, with $a(x) = 1$ for the case of a parallel duct.

2.2   Governing equations

Inside the duct I assume that there is a viscous perfect gas which obeys the Navier Stokes equations [20]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.1a}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \sigma, \tag{2.1b}$$

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left(\mu^B - \frac{2}{3}\mu \right) \delta_{ij} \nabla \cdot \mathbf{u}, \tag{2.1c}$$

$$\rho \frac{DT}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (\kappa \nabla T) + \sigma_{ij} \frac{\partial u_i}{\partial x_j}, \tag{2.1d}$$

$$T = \frac{p}{(\gamma-1)\rho}, \tag{2.1e}$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the convective derivative and $\gamma = c_p^*/c_v^*$ is the ratio of specific heats, typically 1.4 for air. Equation (2.1a) is the continuity equation and corresponds to conservation of mass, equation (2.1b) is the momentum equation and corresponds to Newton's 2nd law for the fluid, equation (2.1d) is the energy equation corresponding to the 1st law of thermodynamics and finally equation (2.1e) is the equation of state for a perfect gas.
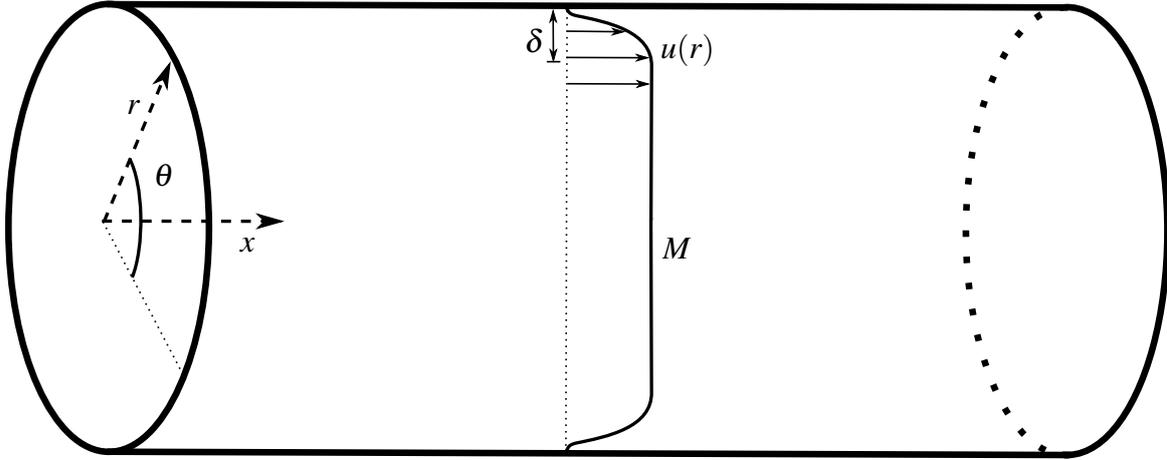
Fig. 2.1 Diagram of the duct.

I assume that the viscosities and thermal conductivity depend linearly on the temperature and are independent of pressure (Prangsma, Alberga & Beenakker) [26], so that I can write

$$\mu = \frac{T}{T_0 \mathrm{Re}}, \quad \mu^B = \frac{T}{T_0 \mathrm{Re}} \frac{\mu_0^{B^*}}{\mu_0^*}, \quad \kappa = \frac{T}{T_0 \mathrm{PrRe}}, \tag{2.2}$$

where T_0 is a nondimensional reference value for the temperature, i.e. the temperature at $r = x = 0$. We also have three nondimensional numbers, the first $Re = c_0^* l^* \rho_0^* / \mu_0^*$ is the Reynolds number, defined with respect to the sound speed and radius of the duct, and is a measure of the inertial forces against the viscous forces. For the case of an aeroengine the Reynolds number is typically very large, in the range of 10^6 to 10^8. Another nondimensional number that appears is the Prandtl number $Pr = \mu_0^* c_p^* / \kappa_0^*$ which is a balance of the rates of viscous diffusion against thermal diffusion, for air it is typically ~ 0.7. The last nondimensional number $\frac{\mu_0^{B^*}}{\mu_0^*}$ is a balance of the bulk viscosity against the dynamic viscosity.

We now have the problem set up as in figure 2.1 where x is the streamwise distance, r the radial distance, and $\delta^2 = \frac{1}{Re}$ will be a small parameter that controls the thickness of the viscous boundary layer.

To solve for the acoustics I will decompose the flow into steady mean flow terms and unsteady acoustic perturbations. e.g. $u = U + \tilde{u}$. The equations (2.1) can then be expanded in terms of the amplitude of the acoustics $|\tilde{p}| \sim \varepsilon$ and the small parameter δ. However before we do this we must first write down our boundary conditions then solve for the steady mean flow, this is done in the following sections.

2.2.1   Boundary conditions

Given the above equations, we now just need the boundary and initial conditions to close the problem so that we can begin solving for the acoustics.

For the initial conditions we will assume our mean flow is steady and that there are no acoustics for $t < 0$. This will only be used when inverting the Fourier transforms of our acoustic quantities, however it will be shown that it has important implications for the stability of the acoustics due to the requirement of causality.

For the boundary conditions we first consider the boundary conditions that apply to the mean flow. At the wall of the duct we will have the no-slip condition

$$\mathbf{u} = 0 \tag{2.3}$$

due to viscosity. This is what causes the boundary layer at the wall of the duct as the effect of viscosity must be considered near the wall to enforce the no-slip condition.

We will also assume that the air inside the duct is in thermal equilibrium with the walls of the duct. This is equivalent to

$$\mathbf{n} \cdot \nabla T = 0 \tag{2.4}$$

at the wall of the duct.

Now for the acoustic terms we first require that on the centreline of the duct, at $r = 0$, all terms are regular. Then, at the wall of the duct we will again require no-slip (2.3) for the streamwise and azimuthal components of the acoustic velocity. However for the normal component we have the impedance condition for the acoustic lining

$$\tilde{p} = Z(\omega)\tilde{v}. \tag{2.5}$$

This is a linear relationship, dependent on the frequency of the mode, between the acoustic pressure and acoustic normal velocity. Note that the dependence on the frequency means that this condition must be applied in Fourier space.

Lastly we require a boundary condition on the acoustic temperature fluctuations at the wall. Here I will assume

$$\tilde{T} = 0 \tag{2.6}$$

at the wall. This is an assumption that the walls of the duct have a far higher thermal capacity than the air within the duct, so that whatever the heat that flows in or out of the duct walls, they remain the same temperature.

2.3   Outer mean flow

2.3.1   Parallel duct

In the case of the parallel duct the mean flow away from the wall of the duct will be assumed to be parallel, uniform and inviscid. This means that all mean flow terms will be constant and the mean flow radial velocity will vanish. Also given that our reference values are taken at the centreline of the duct at $x = 0$ we will have that the streamwise velocity is the Mach number, and the density is one:

$$U = M \quad , \quad V = 0 \quad , \quad \rho = 1 \quad , \quad P = \frac{1}{\gamma} \quad , \quad T = \frac{1}{\gamma - 1}. \tag{2.7}$$

Note that here I am considering the case of a duct with no swirl, that is $W = \mathbf{u} \cdot \hat{\theta} = 0$.

2.3.2   Slowly varying duct

For the case of a slowly varying duct the integrated Navier Stokes equations can be used to derive an implicit equation for the flow Mach number in terms of the duct cross-sectional area. Given the Mach number all other mean flow terms can then be easily computed.

To derive the Mach number we begin by considering the radius of the duct $a(x)$ to be varying slowly and will ignore any time derivatives as we are looking for a steady mean flow. As the Reynolds number is very large and I am assuming that gradients in the centre of the duct are order one, the viscous terms may be ignored and we can consider the steady inviscid Euler equations

$$\nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.8a}$$

$$\rho (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p, \tag{2.8b}$$

$$\rho \mathbf{u} \cdot \nabla T = \mathbf{u} \cdot \nabla p, \tag{2.8c}$$

$$T = \frac{\gamma p}{(\gamma - 1)\rho}. \tag{2.8d}$$

Integrating the continuity equation (2.8a) over a thin slice of the duct as shown in figure 2.2, with the normal $\mathbf{n}$ pointing outwards. We can then use the divergence theorem which gives
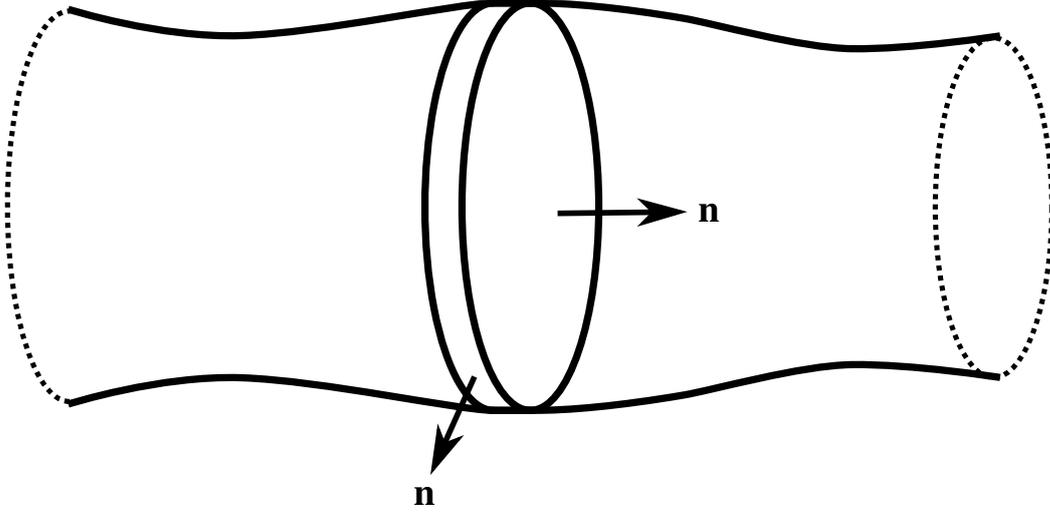
Fig. 2.2 Thin slice of the duct over which we integrate to solve for the outer slowly varying mean flow, with normals **n** in the directions shown.

$$0 = \int_V \nabla \cdot (\rho \mathbf{u}) dV = \int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} dS.$$

But $\mathbf{u} \cdot \mathbf{n} = 0$ on the walls by the no-penetration condition, so

$$\rho U A = \dot{m}, \tag{2.9}$$

where $A(x) = \pi a(x)^2$ is the area of the duct, $\dot{m}$ is a constant and U and ρ are independent of r. I then integrate the energy over the same region and again use the divergence theorem along with the continuity equation, momentum equation (2.8b) and energy equation (2.8c) to find another conserved quantity

$$\int_{\partial V} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho T \right) \mathbf{u} \cdot \mathbf{n} dS = \int_V \frac{|\mathbf{u}|^2}{2} \nabla \cdot (\rho \mathbf{u}) + \frac{\rho}{2} \mathbf{u} \cdot \nabla(|\mathbf{u}|^2) + T \nabla \cdot (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla T dV$$

$$= \int_V 0 - \mathbf{u} \cdot \nabla p + 0 + \mathbf{u} \cdot \nabla p dV = 0.$$

Again using the no penetration condition on the walls of the duct, we then have

$$\left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho T \right) U A = E, \tag{2.10}$$

where E is a constant. Since I am assuming that the duct varies slowly, we know that the radial velocity must be much smaller than the streamwise velocity, so to leading order we can approximate $|\mathbf{u}|^2 \approx U^2$. Lastly I consider the streamwise momentum equation (2.8b). Using the assumptions that U is independent of r to leading order and that the radial velocity is negligible to leading order, this simplifies to

$$\rho U \frac{dU}{dx} = -\frac{dP}{dx}. \tag{2.11}$$

Now we can use the equation of state for a perfect gas (2.8d) along with the conserved quantities (2.9) and (2.10) to derive an expression for the pressure in terms of U and A;

$$P = \frac{(\gamma - 1)\rho \mathcal{T}}{\gamma} = \left(\frac{E}{UA} - \frac{\dot{m}U}{2A} \right) \frac{(\gamma - 1)}{\gamma}.$$

Substituting this into (2.11) and using the conserved quantity (2.9) to substitute for the density we have an equation involving U and A only

$$\frac{\dot{m}}{A} \frac{dU}{dx} = -\frac{(\gamma - 1)}{\gamma} \frac{d}{dx} \left(\frac{E}{UA} - \frac{\dot{m}U}{2A} \right),$$

which can be rearranged to

$$\frac{dU}{dx} \left(\frac{\dot{m}(\gamma + 1)}{2\gamma} - \frac{E(\gamma - 1)}{\gamma U^2} \right) = \frac{d\ln A}{dx} \left(\frac{E}{U} - \frac{\dot{m}U}{2} \right) \frac{(\gamma - 1)}{\gamma}. \tag{2.12}$$

Now we introduce the Mach number $M = \frac{U}{c_s}$ and the speed of sound for a perfect gas $c_s^2 = \frac{\gamma P}{\rho}$. We can then use the equation of state (2.8d) and the conserved quantity (2.10) to derive an expression for the speed of sound

$$c_s^2 = (\gamma - 1)\mathcal{T} = (\gamma - 1) \left(\frac{E}{\dot{m}} - \frac{1}{2}U^2 \right) = (\gamma - 1) \left(\frac{E}{\dot{m}} - \frac{1}{2}M^2 c_s^2 \right). \tag{2.13}$$

This can then be rearranged to give an expression for c_s in terms of M

$$c_s = \sqrt{\frac{(\gamma - 1)E}{\dot{m}(1 + \frac{(\gamma - 1)}{2}M^2)}}.$$

Using this result it is then possible to write the streamwise velocity in terms of M, since $U = Mc_s$. We can then evaluate $\frac{dU}{dx}$ in terms of M

$$\frac{dU}{dx} = \left(1 + M\frac{-\frac{1}{2}\frac{(\gamma-1)}{2}2M}{1 + \frac{(\gamma-1)}{2}M^2}\right)\frac{dM}{dx}c_s = \left(\frac{1}{1 + \frac{(\gamma-1)}{2}M^2}\right)\frac{dM}{dx}c_s. \tag{2.14}$$

We can now substitute this into (2.12) and use (2.13) to cancel the $\frac{E}{\dot{m}}$ terms. This gives the following differential equation for M and A;

$$\frac{d\ln A}{dx} = -\frac{dM}{dx}\left[\frac{1 - M^2}{M(1 + \frac{(\gamma-1)}{2}M^2)}\right]. \tag{2.15}$$

This first order differential equation can be integrated directly, using the initial condition from the nondimensionalisation that $A(x) = \pi$ and $M = M_0$ when $x = 0$;

$$A = \frac{\pi M_0}{M}\left[\frac{1 + \frac{(\gamma-1)}{2}M^2}{1 + \frac{(\gamma-1)}{2}M_0^2}\right]^{\frac{(\gamma+1)}{2(\gamma-1)}}. \tag{2.16}$$

This is an implicit equation for M in terms of A. For any sufficiently large A there are two roots, one corresponding to a subsonic flow ($M < 1$) and the other to a supersonic flow ($M > 1$). However if A is too small it can result in there being either one solution ($M = 1$ the sonic point) or no solutions (the flow is choked). This can be seen in figure 2.3. In this thesis I will deal with cases where $A(x)$ is such that two solutions exist and I will use the lower branch subsonic solution as that is the relevant regime for aeroengines.

Given M we can now compute all other mean flow quantities. The initial conditions for ρ and c_s from the nondimensionalisation allow us to evaluate $\dot{m}$ and E, which gives:

$$\rho = \left[\frac{1 + \frac{(\gamma-1)}{2}M_0^2}{1 + \frac{(\gamma-1)}{2}M^2}\right]^{\frac{1}{(\gamma-1)}}, \quad c_s^2 = \left[\frac{1 + \frac{(\gamma-1)}{2}M_0^2}{1 + \frac{(\gamma-1)}{2}M^2}\right], \tag{2.17}$$

and the other mean flow terms are then given by:

$$P = \frac{1}{\gamma}\rho^\gamma, \quad \mathcal{T} = \frac{\rho^{\gamma-1}}{\gamma - 1}, \quad U = M\left(\frac{\pi M_0}{AM}\right)^{\frac{(\gamma-1)}{(\gamma+1)}} = Mc_s. \tag{2.18}$$
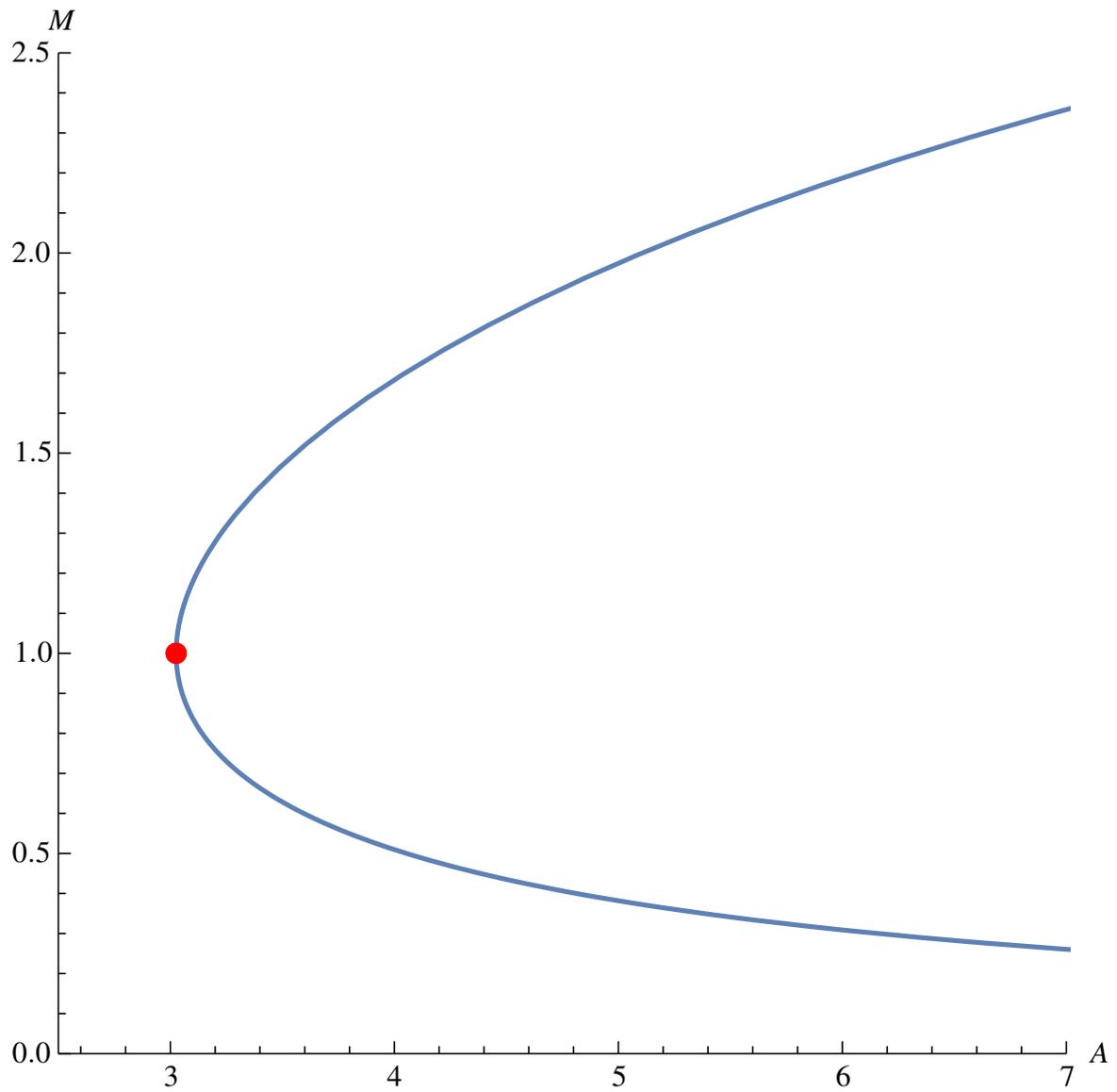
Fig. 2.3 Mach number of the flow in a slowly varying duct against the cross sectional area of the duct, the sonic point is highlighted in red.

While to leading order the radial velocity is zero, we can use the continuity equation (2.8a) to evaluate its first order correction. This gives

$$V = \frac{-r(\rho U)_x}{2\rho} = \frac{a'(x)Ur}{a(x)}, \tag{2.19}$$

which is small, since $|a'(x)| \ll 1$, and is linear in r. We now have all the mean flow terms away from the wall of the duct. If $a'(x) \sim \mathcal{O}(\varepsilon_k)$ and $a''(x) \sim \mathcal{O}(\varepsilon_k^2)$ then the first mean flow correction terms will appear at $\mathcal{O}(\varepsilon_k^2)$ as this is the scale of the terms in the governing equations that we have ignored.

2.4   Boundary layer flow

The mean flow calculated above does not obey the no-slip condition at the wall of the duct. This means that it must be matched to a viscous boundary layer solution near the wall. To derive the boundary layer profile I first consider the scalings near the wall to derive the boundary layer equations, then use several transformations to map to the parallel incompressible case where a similarity solution for the boundary layer profile can be found.

2.4.1   Boundary layer scalings

I begin by assuming that the boundary layer profile is steady. Near the wall it is viscosity that enforces the no-slip condition, so to find the boundary layer thickness I consider the balance of inertial and viscosity

$$\rho(uu_x + vu_r) \sim (\mu u_r)_r. \tag{2.20}$$

If we consider the scaling of each term, assuming that the duct opening is at $x = 0$ so that our streamwise lengthscale is just x, this gives

$$\delta_L \sim \sqrt{\frac{x}{MRe}} = \delta\sqrt{\frac{x}{M}}, \tag{2.21}$$

where δ_L is the radial lengthscale, i.e. the thickness of the boundary layer. Now from the outer uniform mean flow solution we know that $u \sim \mathcal{O}(1)$ and $v \ll u$. We now also have inside the boundary layer $\frac{\partial}{\partial r} \sim \mathcal{O}\left(\frac{1}{\delta}\right)$, $\frac{\partial}{\partial x} \sim \mathcal{O}(1)$ and $\mu \sim \mathcal{O}(\delta^2)$. Using the continuity equation (2.1a) we also have the following scaling:

$$\frac{\partial(\rho u)}{\partial x} \sim \frac{\partial(\rho v)}{\partial r}.$$

This means that at leading order in δ the Navier Stokes equation (2.1) simplify to the boundary layer equations:

$$(\rho u)_x + (\rho v)_r = 0, \tag{2.22a}$$

$$\rho(uu_x + vu_r) = -p_x + (\mu u_r)_r, \tag{2.22b}$$

$$0 = -p_r, \tag{2.22c}$$

$$\rho(uT_x + vT_r) = up_x + vp_r + (\kappa T_r)_r + \mu u_r^2. \tag{2.22d}$$

The radial momentum equation (2.22c) means that the pressure will not vary in the normal direction within the boundary layer and it can be found by matching to the outer mean flow solution (2.18), so $p = P$.

Turbulent boundary layer profiles

It should be noted that the boundary layer equations above (2.22) are valid for laminar boundary layer profiles. Often in aeroacoustics it is desirable to use a turbulent boundary model that incorporates extra terms to try to include the effects of turbulence in the boundary layer profile. While all the results in this thesis are presented using the compressible Blasius boundary layer derived below, the mathematics are valid for any steady boundary layer profile provided it is in thermal equilibrium with the boundary and has no azimuthal component. In chapter 4 when simplifying the expressions using the boundary layer equations I will use the boundary layer equations (2.23) below. These include arbitrary extra terms which could be used for a turbulent boundary layer model

$$(\rho u)_x + (\rho v)_r = 0, \tag{2.23a}$$

$$\rho(uu_x + vu_r) = -p_x + (\mu u_r)_r + \mu F_{rr}(x, r), \tag{2.23b}$$

$$p_r = 0, \tag{2.23c}$$

$$\rho(uT_x + vT_r) = up_x + vp_r + (\kappa T_r)_r + \mu u_r^2 + \mu G_{rr}(x, r), \tag{2.23d}$$

the arbitrary functions F and G are used to model the 'apparent' or 'Reynolds' stress for a turbulent boundary layer model [31].

2.4.2   Non-parallel boundary layer

To solve the boundary layer mean flow in the case of a slowly varying duct we first map to the case of a parallel duct. To do this we begin by introducing y such that $r = a(x) - y$. This means that the wall of the duct is at $y = 0$ and we get the following rules for the partial

derivatives:

$$\frac{\partial}{\partial r} \to -\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + a'\frac{\partial}{\partial y}.$$

We also let $v = -\hat{v} + a'u$ to match with the mean flow V at $r = a$, this transformation also means the advective terms $uu_x + vv_r$ simplify as the $a'(x)$ terms cancel. Using these transformations, the equation of mass conservation now gives

$$(\rho u)_x + (\rho v)_r = 0 \implies (\rho u)_x + (\rho \hat{v})_y = 0,$$

the streamwise momentum equation gives

$$\rho(uu_x + vu_r) = -P_x + (\mu u_r)_r \implies \rho(uu_x + \hat{v}u_y) = -p_x + (\mu u_y)_y,$$

and the energy equation becomes

$$\rho(uT_x + \hat{v}T_y) = uP_x + (\kappa T_y)_y + \mu u_y^2.$$

Now we write $\hat{v} = v$, so we have the same boundary layer equations as before, except that the boundary conditions are now applied at $y = 0$

$$(\rho u)_x + (\rho v)_y = 0, \tag{2.24a}$$

$$\rho(uu_x + vu_y) = -P_x + (\mu u_y)_y, \tag{2.24b}$$

$$p_y = 0, \tag{2.24c}$$

$$\rho(uT_x + vT_y) = uP_x + (\kappa T_y)_y + \mu u_y^2. \tag{2.24d}$$

To solve for the boundary layer profiles we now map, using the Illingworth-Stewartson transformation [31], to the incompressible case by writing

$$\hat{y} = c_s \int_0^y \rho(x, y')dy', \tag{2.25a}$$

$$\hat{x} = \int_0^x \gamma P(x')c_s(x')dx', \tag{2.25b}$$

where P and c_s are the pressure and speed of sound respectively for the outer mean flow solution. This transformation gives $\frac{\partial \hat{y}}{\partial y} = c_s\rho$, $\frac{\partial \hat{x}}{\partial x} = \gamma Pc_s$ and $\frac{\partial \hat{x}}{\partial y} = 0$.

Flow profile

To solve for the mean flow I begin by solving for the velocity profile using equations (2.24a) and (2.24b). To do this I introduce a streamfunction ψ, where $\rho u = \psi_y$ and $\rho v = -\psi_x$ so that the continuity equation (2.24a) is satisfied. Now $\psi(x,y) = \hat{\psi}(\hat{x}, \hat{y})$, so under the transformation $(x, y) \rightarrow (\hat{x}, \hat{y})$ we get

$$\rho u = \hat{\psi}_{\hat{y}} c_s \rho \implies u = c_s \hat{\psi}_{\hat{y}} = c_s \hat{u}$$

and

$$\rho v = -\hat{\psi}_{\hat{x}} \gamma P c_s - \hat{\psi}_{\hat{y}} \frac{\partial \hat{y}}{\partial x}.$$

We can then write down the derivatives of the streamwise velocity

$$u_x = \gamma P c_s (c_{s\hat{x}} \hat{\psi}_{\hat{y}} + c_s \hat{\psi}_{\hat{y}\hat{x}}) + c_s \hat{\psi}_{\hat{y}\hat{y}} \hat{y}_x,$$
$$u_y = \rho c_s^2 \hat{\psi}_{\hat{y}\hat{y}},$$

which then gives the advective term from the streamwise momentum equation

$$u u_x + v u_y = \gamma P c_s^3 (\hat{\psi}_{\hat{y}}^2 \frac{c_{s\hat{x}}}{c_s} + \hat{\psi}_{\hat{y}\hat{x}} \hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}} \hat{\psi}_{\hat{x}}).$$

If we now consider the viscous term in the streamwise momentum equation

$$(\mu u_y) = \delta^2 (\gamma - 1) T \rho c_s^2 \hat{\psi}_{\hat{y}\hat{y}} = \delta^2 \gamma P c_s^2 \hat{\psi}_{\hat{y}\hat{y}},$$

so

$$\frac{1}{\rho} (\mu u_y)_y = \delta^2 \gamma P c_s^3 \hat{\psi}_{\hat{y}\hat{y}\hat{y}}.$$

Putting all these terms together to form the streamwise momentum equation we now have

$$\hat{\psi}_{\hat{y}\hat{x}} \hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}} \hat{\psi}_{\hat{x}} - \delta^2 \hat{\psi}_{\hat{y}\hat{y}\hat{y}} = -\hat{\psi}_{\hat{y}}^2 \frac{c_{s\hat{x}}}{c_s} - \frac{P_{\hat{x}}}{\rho c_s^2}. \tag{2.26}$$

This can also be written as:

$$\hat{\psi}_{\hat{y}\hat{x}} \hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}} \hat{\psi}_{\hat{x}} - \delta^2 \hat{\psi}_{\hat{y}\hat{y}\hat{y}} = \frac{\frac{1}{2} u^2 + T}{\frac{1}{\gamma - 1} + \frac{1}{2} M_0^2} M M_{\hat{x}}, \tag{2.27}$$

as shown in appendix A.1. The right hand side of this equation contains an M_x term which from the implicit outer mean flow solution must be small. This means that at leading order the right hand side is negligible and we only need to solve for the left hand side. If we wanted to satisfy the full equation we could use an asymptotic expansion of $\hat{\psi}$ to find the first order in a' correction terms, this would result in a linear PDE for the correction terms that is forced by the leading order solution. However here I will only use the leading order solution for simplicity.

The left hand side of this equation has the standard form for a Blasius boundary layer streamfunction [31], so we can try a similarity solution of the form

$$\hat{\psi} = \delta\sqrt{M\hat{x}}f(\hat{\zeta}), \tag{2.28}$$

with

$$\hat{\zeta} = \frac{\hat{y}}{\delta}\sqrt{\frac{M}{\hat{x}}}. \tag{2.29}$$

Using this ansatz, and noting that $\hat{x}$ derivatives of M are negligible at leading order, the equation then becomes the Blasius equation

$$f''' + \frac{ff''}{2} = 0. \tag{2.30}$$

This equation corresponds to the incompressible Blasius boundary layer in the case $\hat{x} = x$ and $\hat{y} = y$. So the functions $\hat{x}(x)$ and $\hat{y}(x,y)$ map the incompressible solution to the compressible solution that we are interested in.

To satisfy the no-slip condition at the wall we have $u = 0$ and $v = 0$ at $\hat{\zeta} = 0$, and matching to the outer mean flow solution gives $u \to U$ or in the transformed variables $\hat{u} \to M$ as $\hat{\zeta} \to \infty$. These then imply the following boundary conditions on $f(\hat{\zeta})$

$$f = f' = 0 \quad \text{at} \quad \hat{\zeta} = 0 \qquad \text{and} \qquad f' \to 1 \quad \text{as} \quad \hat{\zeta} \to \infty.$$

The equation can then be solved numerically using the shooting method by specifying $f''(0)$ and solving the initial value problem, then updating the value of $f''(0)$ and repeating until the boundary condition $f' \to 1$ as $\hat{\zeta} \to \infty$ is satisfied.

Finally at leading order we have

$$u = c_s\hat{u} = c_sMf'(\hat{\zeta}). \tag{2.31}$$

Temperature profile

Now that I have the boundary layer velocity profile I just need the boundary layer temperature profile to be able to evaluate all boundary layer mean flow terms. To derive the temperature profile I now consider the energy equation (2.24d). Using the above solution for the flow velocities, we can again evaluate each term in the equation. Firstly the convective terms

$$uT_x + vT_y = \gamma P c_s^2 (\hat{\psi}_{\hat{y}} T_{\hat{x}} - \hat{\psi}_{\hat{x}} T_{\hat{y}}),$$

the pressure term

$$\frac{uP_x}{\rho} = \frac{\gamma P c_s^2 \hat{\psi}_{\hat{y}} P_{\hat{x}}}{\rho},$$

the thermal diffusion term

$$\frac{1}{\rho}(\kappa T_y)_y = \frac{\delta^2 \gamma P c_s^2}{Pr} T_{\hat{y}\hat{y}},$$

and lastly the viscous dissipation term

$$\frac{\mu}{\rho} u_y^2 = \delta^2 \gamma P c_s^4 \hat{\psi}_{\hat{y}\hat{y}}^2.$$

Putting this all together the transformed energy equation is

$$\hat{\psi}_{\hat{y}} T_{\hat{x}} - \hat{\psi}_{\hat{x}} T_{\hat{y}} - \frac{\delta^2}{Pr} T_{\hat{y}\hat{y}} - \delta^2 c_s^2 \hat{\psi}_{\hat{y}\hat{y}}^2 = \hat{\psi}_{\hat{y}} \frac{P_{\hat{x}}}{\rho}. \tag{2.32}$$

We now let $T = c_s^2 \tau$, so $T_{\hat{y}} = c_s^2 \tau_{\hat{y}}$ and $T_{\hat{x}} = c_s^2 \tau_{\hat{x}} + 2c_s c_{s\hat{x}} \tau$. This transforms the equation to

$$\hat{\psi}_{\hat{y}} \tau_{\hat{x}} - \hat{\psi}_{\hat{x}} \tau_{\hat{y}} - \frac{\delta^2}{Pr} \tau_{\hat{y}\hat{y}} - \delta^2 \hat{\psi}_{\hat{y}\hat{y}}^2 = \hat{\psi}_{\hat{y}} \frac{P_{\hat{x}}}{\rho} - 2c_s c_{s\hat{x}} \tau \hat{\psi}_{\hat{y}}, \tag{2.33}$$

which rearranges to

$$\hat{\psi}_{\hat{y}} \tau_{\hat{x}} - \hat{\psi}_{\hat{x}} \tau_{\hat{y}} - \frac{\delta^2}{Pr} \tau_{\hat{y}\hat{y}} - \delta^2 \hat{\psi}_{\hat{y}\hat{y}}^2 = 0, \tag{2.34}$$

as shown in appendix A.2. We can now use the leading order solution for $\hat{\psi}$ from above and assume that $\tau = \tau(\hat{\zeta})$ to solve for the leading order solution. This results in a second order ODE for $\tau(\hat{\zeta})$

$$\frac{\tau_{\hat{\zeta}\hat{\zeta}}}{Pr} + \frac{f\tau_{\hat{\zeta}}}{2} + M^2 f_{\hat{\zeta}\hat{\zeta}}^2 = 0. \tag{2.35}$$

Given that $\tau \to \frac{1}{\gamma-1}$ as $\hat{\zeta} \to \infty$ we use the following change of variables

$$\tau = \frac{1}{\gamma-1} + \frac{M^2}{2}\hat{\tau},$$

equation (2.35) then becomes:

$$\hat{\tau}'' + \frac{Prf\hat{\tau}'}{2} = -2Prf''^2. \qquad (2.36)$$

We can now use an integrating factor $\exp\left(\int \frac{Prf}{2}\, d\hat{\zeta}\right)$ to solve for $\hat{\tau}$. However we can first use the equation for f (2.30) to get a nicer form of the integrating factor. We have

$$\frac{f'''}{f''} = -\frac{f}{2},$$

so

$$\int \frac{f}{2}\, d\hat{\zeta} = -\ln(f'').$$

Using this in equation (2.36) then gives

$$\left(\frac{\hat{\tau}'}{(f'')^{Pr}}\right)' = -2Pr(f'')^{2-Pr},$$

which can be rearranged to a first order differential equation for $\hat{\tau}$,

$$\hat{\tau}' = -2Pr(f'')^{Pr}\int_0^{\hat{\zeta}} (f''(q))^{2-Pr}\, dq, \qquad (2.37)$$

with $\hat{\tau} \to 0$ as $\hat{\zeta} \to \infty$. This equation can easily be solved numerically by integrating the equation from $\hat{\zeta} = 0$ then subtracting the resulting value of $\hat{\tau}(\infty)$ to set the arbitrary integration constant, so that the correct boundary condition is satisfied.

At leading order we now have

$$T = \frac{c_s^2}{\gamma-1} + \frac{M^2 c_s^2}{2}\hat{\tau}(\hat{\zeta}). \qquad (2.38)$$

We can now also implicitly invert the $\hat{x}$, $\hat{y}$ transformation using the integrals (2.25).

2.4.3 Parallel duct boundary layer profile

In the parallel duct case the above boundary layer profile solutions can be simplified. From the outer mean flow solution for the parallel duct, we have $c_s = 1$ and $P = \frac{1}{\gamma}$. This then gives

$$\hat{y} = \int_0^y \rho(x, y') dy', \tag{2.39a}$$

$$\hat{x} = x. \tag{2.39b}$$

So we now only have to transform the y coordinate, and our boundary layer profile is

$$u = M f'(\hat{\zeta}), \tag{2.40}$$

$$T = \frac{1}{\gamma - 1} + \frac{M^2}{2} \hat{\tau}(\hat{\zeta}). \tag{2.41}$$

Since the pressure is constant through the boundary layer and is $\frac{1}{\gamma}$ in the outer region, using the equation of state (2.1e) we have

$$\rho = \frac{1}{(\gamma - 1)T} \tag{2.42}$$

inside the boundary layer. Also since $a' = 0$ the normal velocity simplifies

$$\rho v = -\gamma P c_s \hat{\psi}_{\hat{x}} - \hat{\psi}_{\hat{y}} \hat{y}_x = -\gamma P c_s \frac{\delta}{2} \sqrt{\frac{M}{x}} (f - \hat{\zeta} f') - M f' \hat{y}_x,$$

but $\hat{y}_x = \int_0^y \frac{\partial \rho}{\partial x} dy$ and $\frac{\partial \rho}{\partial x}$ is only non-zero inside the boundary layer, i.e. a region of width δ so $\hat{y}_x \sim \mathcal{O}(\delta)$. This means that for the parallel duct we have

$$v = \mathcal{O}(\delta). \tag{2.43}$$

Figure 2.4 shows a plot of the boundary layer profiles for $M = 0.7$.

2.5 Fourier transform convention

To solve for the acoustics I will take Fourier transforms in x and t and, for periodicity, a Fourier series in θ. In this thesis I will use the following conventions when taking Fourier
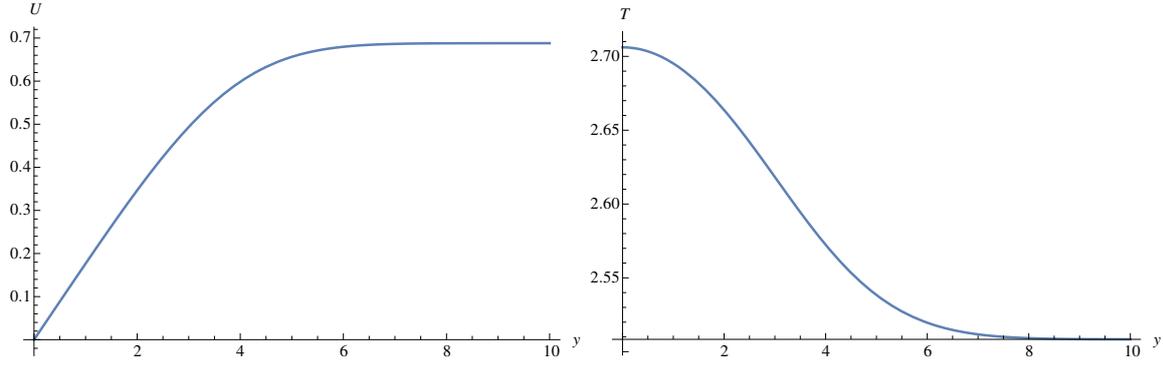
Fig. 2.4 Plot of the boundary layer streamwise velocity and temperature profiles

transforms/series. For the ω-time transform I use the following convention

$$\tilde{f}(\omega) = \mathscr{F}_\omega[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt, \tag{2.44}$$

which then gives the inverse transform

$$f(t) = \mathscr{F}_t^{-1}[\tilde{f}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt. \tag{2.45}$$

Whereas for the k-spatial transform I use the opposite sign convention

$$\tilde{f}(k) = \mathscr{F}_k[f(x)] = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx, \tag{2.46}$$

and thus

$$f(x) = \mathscr{F}_x^{-1}[\tilde{f}(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{2.47}$$

Finally for the θ Fourier series I use the same sign convention as for the k-spatial transform

$$\tilde{f}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{im\theta}, \tag{2.48}$$

so that we have

$$f(\theta) = \sum_{-\infty}^{\infty} \tilde{f}(m)e^{-im\theta}. \tag{2.49}$$

Chapter 3

Weakly nonlinear parallel acoustics

3.1 Introduction

In this chapter I will investigate the weakly nonlinear acoustics in a lined duct with parallel boundary layer flow. The assumption of a parallel boundary layer flow is a standard assumption in the field of duct acoustics [24, 5]. We assume that we are far enough downstream of the leading edge of the duct such that streamwise variations in the mean flow are negligible. This means that we can use the parallel outer mean flow §2.3.1 and the parallel boundary layer mean flow §2.4.3 at some fixed value downstream.

To solve for the weakly nonlinear acoustics in the duct I begin with the leading order linear acoustics solution developed by Brambley [4]. This takes into account the effects of both shear and viscosity on the acoustics inside the boundary layer. I will then calculate the weakly nonlinear acoustics due to the leading order solution and show that in certain cases a surprisingly large, rapidly varying acoustic streaming mode can propagate into the centre of the duct.

3.2 Mean flow

As discussed in the introduction, in this chapter I consider a parallel duct with a boundary layer of thickness δ to be constant, independent of x. I assume that the nondimensionalised radius of the duct is $r = 1$ so the parallel outer mean flow §2.3.1 can be used. I then set the boundary layer thickness

$$\xi \delta^2 = 1/\text{Re}, \tag{3.1}$$

where $\xi \sim \mathcal{O}(1)$ is a parameter adjusting how well developed the boundary layer is, and $\xi = 0$ gives an inviscid boundary layer. Any boundary layer profile could be used for what follows, provided it is independent of t, x and θ and is in thermal equilibrium with the boundary, $T_r(1) = 0$. For the results given here, a compressible Blasius boundary layer is used as was derived in §2.4.3, with the downstream distance, x, fixed so that $\delta_L = \delta$. This is equivalent to setting $x = \frac{M}{\xi}$ fixed and assuming streamwise derivatives of the mean flow terms are negligible.

3.3 Acoustic perturbations

We now consider small monochromatic perturbations to the mean flow of magnitude $\varepsilon \ll \delta$ that are the real part of terms with dependence $\exp\{i(\omega t - kx - m\theta)\}$. We will write, for example, the total temperature as $T + \tilde{T}$, where $T(r)$ is the mean flow value and $\tilde{T}$ is the small harmonic perturbation.

Outside the boundary layer (i.e. within the duct away from the walls) we assume gradients are not large, so that the viscous terms, which are $\mathcal{O}(1/\mathrm{Re}) = \mathcal{O}(\delta^2)$ from (2.2), can be neglected at leading order. This means that at leading order we can treat the flow outside the boundary layer as inviscid and can use the results from §2.3.1 for the mean flow terms with $\mathbf{u} = (M + \tilde{u}_O, \tilde{v}_O, \tilde{w}_O)$, $p = 1/\gamma + \tilde{p}_O$ and $T = 1/(\gamma - 1) + \tilde{T}_O$, where all the acoustic perturbations, $\tilde{u}_O$, $\tilde{v}_O$, $\tilde{w}_O$, $\tilde{p}_O$ and $\tilde{T}_O$ are $\mathcal{O}(\varepsilon)$ at leading order.

Inside the boundary layer, we rescale using (3.1), so that

$$r = 1 - \delta y \qquad \text{and} \qquad \mathbf{u} = (u + \tilde{u}, -\delta \tilde{v}, \tilde{w}). \qquad (3.2)$$

Note that the factor of δ in front of $\tilde{v}$ is needed for terms to balance at leading order (see, e.g. [4]), while the minus sign is for convenience so that $\tilde{v}_O$ is positive in the positive r direction while $\tilde{v}$ is positive in the positive y direction. From [4] we know that for the leading order system to be well-posed we require $\tilde{u}$, $\tilde{v}$, $\tilde{\rho}$ and $\tilde{T}$ to be $\mathcal{O}(\varepsilon/\delta)$ and $\tilde{p}$ and $\tilde{w}$ to be $\mathcal{O}(\varepsilon)$

at leading order. This suggests we use the expansion

$$\tilde{u} = \frac{\varepsilon}{\delta}\tilde{u}_1 + \frac{\varepsilon^2}{\delta^2}\tilde{u}_2 + \varepsilon\tilde{u}_3, \tag{3.3a}$$

$$\tilde{v} = \frac{\varepsilon}{\delta}\tilde{v}_1 + \frac{\varepsilon^2}{\delta^2}\tilde{v}_2 + \varepsilon\tilde{v}_3, \tag{3.3b}$$

$$\tilde{w} = \varepsilon\tilde{w}_1 + \frac{\varepsilon^2}{\delta}\tilde{w}_2 + \varepsilon\delta\tilde{w}_3, \tag{3.3c}$$

$$\tilde{T} = \frac{\varepsilon}{\delta}\tilde{T}_1 + \frac{\varepsilon^2}{\delta^2}\tilde{T}_2 + \varepsilon\tilde{T}_3, \tag{3.3d}$$

$$\tilde{p} = \varepsilon\tilde{p}_1 + \frac{\varepsilon^2}{\delta}\tilde{p}_2 + \varepsilon\delta\tilde{p}_3, \tag{3.3e}$$

where the quantities labelled '1' are the leading order (linear) perturbations, quantities labelled '2' are the first order nonlinear correction, and quantities labelled '3' are the first order in δ linear correction (i.e. the first terms to involve mean flow shear). Note that, at leading order within the boundary layer, $-\delta\tilde{v} = \mathcal{O}(\varepsilon)$, the same as outside the boundary layer, but $\tilde{u} = \mathcal{O}(\varepsilon/\delta)$, an amplification of $1/\delta$ compared to $\tilde{u}_O$.

The solution for the leading order linear perturbations was given by Brambley [4] while the solution for the first order in δ linear correction terms has been given by Khamis [15]. In this chapter I will solve for the nonlinear order ε^2 terms. To do this it is necessary to expand the governing equations §(2.1) first to leading order to solve for the leading order linear acoustics and then to order $\mathcal{O}(\varepsilon^2)$ to give a linear system of ODEs for the weakly nonlinear quantities.

3.4   Linear acoustics

In this section I describe the process for solving the governing equations (2.1) with the asymptotic expansion (3.3) for the leading order $\mathcal{O}(\varepsilon)$ linear terms (quantities labelled '1'), reproducing the results of [4]. A similar procedure can be used for the first order linear correction terms (quantities labelled '3'), as was done in [18]. Since the equations that I am working with here are linear, we do not have to take the real parts of the complex exponentials when substituting for the perturbations, and may instead work directly with the complex exponentials, as is usual in acoustics.

Given the assumption that the acoustic perturbations are harmonic this will result in a homogeneous set of linear ODEs in both the outer region and the inner boundary layer region.

3.4.1   Derivation of linear acoustics equations

I begin by considering the leading order linear acoustics. I first consider the equation of state for a perfect gas (2.1e), expanding in powers of ε and δ gives

$$\rho + \varepsilon \tilde{\rho}_1 = \frac{\gamma p}{(\gamma - 1)T} + \varepsilon \left(\frac{\gamma \tilde{p}_1}{(\gamma - 1)T} - \frac{\tilde{T}_1}{(\gamma - 1)T^2} \right) + \mathcal{O}(\varepsilon^2, \varepsilon \delta).$$

From the derivation for the steady mean flow §2.4 we know that the $\mathcal{O}(1)$ steady leading order equation is satisfied, so we can match the terms of order ε to get

$$\tilde{\rho}_1 = \frac{\tilde{p}_1 \gamma}{(\gamma - 1)T} - \frac{\tilde{T}_1}{(\gamma - 1)T^2}. \tag{3.4}$$

This equation is independent of δ and has no derivatives so it will hold in both the outer inviscid region and the inner boundary layer region. Now I consider the continuity equation (2.1a)

$$\rho_t + (\rho u)_x + \frac{1}{r}(r\rho v)_r + \frac{1}{r}(\rho w)_\theta = 0,$$

note that the subscripts here denote differentiation (e.g. $\rho_t = \frac{\partial \rho}{\partial t}$). Considering only the leading order $\mathcal{O}(\varepsilon)$ time harmonic terms this becomes

$$\mathrm{i}(\omega - uk)\tilde{\rho}_1 - \mathrm{i}k\rho \tilde{u}_1 + \frac{1}{r}\rho \tilde{v}_1 + \rho_r \tilde{v}_1 + \rho \tilde{v}_{1r} - \frac{\mathrm{i}m}{r}\rho \tilde{w}_1 = 0.$$

Equation (3.4) can then be used to eliminate $\tilde{\rho}_1$ and (2.1e) to eliminate ρ, this gives the following equation

$$\mathrm{i}(\omega - uk)\gamma \tilde{p}_1 - \frac{\mathrm{i}(\omega - uk)}{T}\tilde{T}_1 - \mathrm{i}k\tilde{u}_1 + \frac{1}{r}\tilde{v}_1 - \frac{T_r}{T}\tilde{v}_1 + \tilde{v}_{1r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 = 0. \tag{3.5}$$

In the outer region the mean flow profile is uniform so this equation simplifies to

$$\mathrm{i}(\omega - Mk)\gamma \tilde{p}_1 - \mathrm{i}(\omega - Mk)(\gamma - 1)\tilde{T}_1 - \mathrm{i}k\tilde{u}_1 + \frac{1}{r}\tilde{v}_1 + \tilde{v}_{1r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 = 0, \tag{3.6}$$

and in the inner region, under the rescaling (3.2) and taking into account the amplification (3.3) of $\tilde{u}$, $\tilde{v}$ and $\tilde{T}$ equation (3.5) simplifies to

$$\mathrm{i}(\omega - uk)\tilde{T}_1 + \mathrm{i}kT\tilde{u}_1 + T_y \tilde{v}_1 - T\tilde{v}_{1y} = 0. \tag{3.7}$$

Now μ is given by equation (2.2) which can be rewritten in terms of δ

$$\mu = \frac{T}{T_0 Re} = \xi \delta^2 (\gamma - 1) T. \tag{3.8}$$

Using this we can now consider the streamwise momentum equation (this is the x-component of equation (2.1b))

$$\rho \left(u_t + u u_x + v u_r + \frac{1}{r} w u_\theta \right) = -p_x + (\sigma_{11})_x + \frac{1}{r}(r\sigma_{12})_r + \frac{1}{r}(\sigma_{13})_\theta, \tag{3.9}$$

the leading order in ε expansion then gives

$$\frac{\mathrm{i}(\omega - uk)}{(\gamma - 1)T}\tilde{u}_1 + \frac{u_r \tilde{v}_1}{(\gamma - 1)T} - \mathrm{i}k\tilde{p}_1 + 2k^2 \mu \tilde{u}_1 + \mathrm{i}k\mu \left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3} \right) \left[-\mathrm{i}k\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 \right]$$
$$+ \frac{\mu m}{r} \left(k\tilde{w}_1 + \frac{m}{r} \right) \tilde{u}_1 - \left(\mu_r (\tilde{u}_{1r} - \mathrm{i}k\tilde{v}_1) \right)_r - \mu \left(\frac{\tilde{u}_{1r}}{r} - \frac{\mathrm{i}k\tilde{v}_1}{r} \right) - u_r \left(\tilde{\mu}_{1r} + \frac{\tilde{\mu}_1}{r} \right) - u_{rr}\tilde{\mu}_1 = 0, \tag{3.10}$$

where $\tilde{\mu} = \frac{\tilde{T}}{T_0 Re} = \xi \delta^2 (\gamma - 1)\tilde{T}$. In the outer region we assume our gradients are $\mathscr{O}(1)$ and given that μ and $\tilde{\mu}$ are $\mathscr{O}(\delta^2)$ this allows the equation to be greatly simplified as the viscous terms may be neglected

$$\mathrm{i}(\omega - Mk)\tilde{u}_1 - \mathrm{i}k\tilde{p}_1 = 0. \tag{3.11}$$

In the inner region from the definition of δ as the thickness of the boundary layer we know that gradients in r are $\mathscr{O}(1/\delta)$. This means that the only viscous terms that will appear at leading order are those with two r-derivatives. We can then write down the leading order equation for the acoustics in the inner region again taking into account the amplification (3.3) and the rescaling (3.2)

$$\mathrm{i}(\omega - uk)\tilde{u}_1 + \tilde{v}_1 u_y - \xi(\gamma - 1)^2 T (T\tilde{u}_{1y} + \tilde{T}_1 u_y)_y = 0. \tag{3.12}$$

If we now consider radial momentum equation

$$\rho \left(v_t + u v_x + v v_r + \frac{1}{r} w v_\theta \right) = \frac{1}{r}\rho w^2 - p_r - \frac{1}{r}\sigma_{33} + (\sigma_{12})_x + \frac{1}{r}(r\sigma_{22})_r + \frac{1}{r}(\sigma_{23})_\theta, \tag{3.13}$$

in the outer region, given the assumption of $\mathscr{O}(1)$ gradients, we can see straight away that the viscous terms may be ignored since, from (3.8), they will be at least $\mathscr{O}(\varepsilon\delta^2)$. If we then

consider the remaining terms it is easy to see that the leading order $\mathcal{O}(\varepsilon)$ balance will give

$$\mathrm{i}(\omega - Mk)\tilde{v}_1 = -\tilde{p}_{1r}. \tag{3.14}$$

Now in the inner boundary layer region we find that the viscous terms can be $\mathcal{O}(\varepsilon)$ due to the large radial gradients. However there is also a term involving the radial gradient of pressure which in the inner region is $\mathcal{O}(\frac{\varepsilon}{\delta})$. Since $\tilde{v}$ is only this order after the rescaling (3.2) we find that this pressure term cannot be balanced and so the equation in the boundary layer is simply

$$\tilde{p}_{1y} = 0. \tag{3.15}$$

If we now consider θ-momentum equation

$$\rho\left(w_t + uw_x + vw_r + \frac{1}{r}ww_\theta\right) = -\frac{1}{r}\rho vw - \frac{1}{r}p_\theta + (\sigma_{13})_x + \frac{1}{r^2}(r^2\sigma_{23})_r + \frac{1}{r}(\sigma_{33})_\theta, \tag{3.16}$$

for the outer solution it is again clear that the viscous terms will not appear at leading order in δ, similarly to the streamwise and radial momentum equations. So the equation becomes

$$\mathrm{i}(\omega - Mk)\tilde{w}_1 = \frac{\mathrm{i}m}{r}\tilde{p}_1, \tag{3.17}$$

while for the inner region the expansion is similar to that for the streamwise momentum equation. Again the only viscous terms that will appear at leading order are those with two r-derivatives. This means that the leading order inner equation is

$$\mathrm{i}(\omega - uk)\tilde{w}_1 - \xi(\gamma - 1)^2 T(T\tilde{w}_{1y})_y = \mathrm{i}m\tilde{p}_1. \tag{3.18}$$

Lastly we consider the energy equation

$$\rho\left(T_t + uT_x + vT_r + \frac{1}{r}wT_\theta\right) = \left(p_t + up_x + vp_r + \frac{1}{r}wp_\theta\right) + (\kappa T_x)_x + \frac{1}{r}(r\kappa T_r)_r + \frac{1}{r^2}(\kappa T_\theta)_\theta$$
$$+ \sigma_{11}u_x + \sigma_{12}(u_r + v_x) + \sigma_{13}\left(\frac{1}{r}u_\theta + w_x\right) + \sigma_{22}v_r + \sigma_{23}\left(\frac{1}{r}v_\theta + r\left(\frac{w}{r}\right)_r\right) + \frac{1}{r}\sigma_{33}(w_\theta + v), \tag{3.19}$$

we can note that $\kappa = \mu/Pr$, so in the outer region we find all κ and μ terms to be negligible. The equation then simplifies to

$$\mathrm{i}(\omega - Mk)\tilde{T}_1 = \mathrm{i}(\omega - Mk)\tilde{p}_1. \tag{3.20}$$

In the inner region, given the amplification (3.3), the pressure terms will be negligible and the only terms that appear from the right hand side are those with two radial derivatives. So we can simplify the equation to get

$$\mathrm{i}(\omega - uk)\tilde{T}_1 + \tilde{v}_1 T_y - \xi(\gamma - 1)^2 T \left[\frac{1}{Pr}(\tilde{T}_1 T)_{yy} + \tilde{T}_1(u_y)^2 + 2Tu_y\tilde{u}_{1y} \right] = 0. \qquad (3.21)$$

We now have a set of five equations for the acoustics in both the inner and outer regions. In the outer region we can notice that equations (3.11), (3.17) and (3.20) are algebraic only and may be substituted easily into equation (3.6) along with equation (3.14). This results in the following second order ODE for the acoustic pressure perturbation in the outer region

$$\tilde{p}_{1rr} + \frac{\tilde{p}_{1r}}{r} + \tilde{p}_1 \left((\omega - Mk)^2 - k^2 - \frac{m^2}{r^2} \right) = 0. \qquad (3.22)$$

For the inner region we can notice that the equations uncouple, as equations (3.15) and (3.18) only involve the acoustic pressure and acoustic azimuthal velocity perturbations while equations (3.7), (3.12) and (3.21) involve only the acoustic radial and streamwise velocity and acoustic temperature pertubations. This gives the following system of second order linear ODEs for the acoustic normal velocity, acoustic streamwise velocity and acoustic temperature

$$\mathcal{L}(\tilde{u}_1, \tilde{v}_1, T_1; \omega, k, m) = \mathbf{0}, \qquad (3.23)$$

where

$$\mathcal{L} = \left\{ \begin{array}{c} \mathrm{i}(\omega - uk)\tilde{T}_1 + T_y\tilde{v}_1 - T\tilde{v}_{1y} + \mathrm{i}kT\tilde{u}_1 \\ \mathrm{i}(\omega - uk)\tilde{u}_1 + \tilde{v}_1 u_y - \xi(\gamma - 1)^2 T (T\tilde{u}_{1y} + \tilde{T}_1 u_y)_y \\ \mathrm{i}(\omega - uk)\tilde{T}_1 + \tilde{v}_1 T_y - \xi(\gamma - 1)^2 T \left[\frac{1}{Pr}(\tilde{T}_1 T)_{yy} + \tilde{T}_1(u_y)^2 + 2Tu_y\tilde{u}_{1y} \right] \end{array} \right\}. \qquad (3.24)$$

From (3.15) the acoustic pressure perturbation is constant within the inner region so may be found by matching to the outer region. The acoustic azimuthal velocity can then be found using equation (3.18).

3.4.2   Outer solution in the duct interior

I have shown in §3.4.1 that at leading order the governing equations in the outer region reduce to (3.22) which is the standard Bessel's equation for acoustics in a duct with cylindrical cross section. Applying the boundary condition that the solution must be regular at $r = 0$, the

solution for the pressure is given by:

$$\tilde{p}_O = CJ_m(\alpha r), \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2, \qquad (3.25)$$

and C is an arbitrary constant. The other quantities are then given in terms of $\tilde{p}_O$ by

$$i(\omega - Mk)\tilde{u}_O - ik\tilde{p}_O = 0, \qquad (3.26a)$$

$$i(\omega - Mk)\tilde{v}_O + \tilde{p}_{Or} = 0, \qquad (3.26b)$$

$$i(\omega - Mk)\tilde{w}_O - im\tilde{p}_O/r = 0, \qquad (3.26c)$$

$$\tilde{T}_O = \tilde{p}_O. \qquad (3.26d)$$

Now that we have all the acoustic quantities we just need to apply a boundary condition at the wall to close the system and obtain a dispersion relation $\omega(k)$. However the above solution is not valid within the boundary layer close to the wall, it does not satisfy the no-slip condition (2.3) or take into account the mean flow boundary layer shear. So we must solve for the acoustics in the inner boundary layer region where we can apply the impedance condition at the wall. We can then match our solutions in an intermediate region to obtain the dispersion relation.

3.4.3 Inner solution in the boundary layer

Inside the boundary layer, the expansion of the governing equations (2.1) at leading order gives the system of equations (3.23)

$$\left\{ \begin{array}{c} i(\omega - uk)\tilde{T}_1 + T_y\tilde{v}_1 - T\tilde{v}_{1y} + ikT\tilde{u}_1 \\ i(\omega - uk)\tilde{u}_1 + \tilde{v}_1 u_y - \xi(\gamma - 1)^2 T(T\tilde{u}_{1y} + \tilde{T}_1 u_y)_y \\ i(\omega - uk)\tilde{T}_1 + \tilde{v}_1 T_y - \xi(\gamma - 1)^2 T \left[\frac{1}{Pr}(\tilde{T}_1 T)_{yy} + \tilde{T}_1(u_y)^2 + 2Tu_y\tilde{u}_{1y} \right] \end{array} \right\} = \mathbf{0}.$$

This is a coupled system of linear homogeneous ODEs in y. It contains terms involving u and T and so takes into account the full mean flow boundary layer profile. Since our boundary layer profile is generally found numerically and this system of equations is not directly integrable we must also solve this system of equations numerically. First however we must consider the boundary conditions that we will apply.

The boundary conditions at the wall ($y = 0$) are those of no slip ($\tilde{u}_1 = 0$), thermal equilibrium ($\tilde{T}_1 = 0$, obtained by assuming the wall has a far higher thermal capacity than the fluid), and the impedance boundary condition $\tilde{p} = Z(\omega)\tilde{v}$. Since the system of equations $\mathscr{L}$ is second order in $\tilde{u}$ and $\tilde{T}$, one further boundary condition on each of $\tilde{u}$ and $\tilde{T}$ is needed,

which is obtained by requiring the inner solution to be compatible with an outer solution as $y \to \infty$. Finding the compatible outer solution to match to the inner solution is considered in the next section.

3.4.4   Matching the outer and inner solutions

Sufficiently far outside the boundary layer for $y \geq Y \gg 1$, the gradients of the mean flow quantities vanish and the mean flow quantities attain their uniform mean flow values. Hence, for $y \geq Y$ the system of ODEs (3.23) decouples and becomes

$$\mathscr{L}(\tilde{u}_1, \tilde{v}_1, T_1; \omega, k) = \left\{ \begin{array}{c} \eta_\infty^2 \xi \tilde{T}_1 - \frac{1}{(\gamma-1)} \tilde{v}_{1y} + \frac{ik}{(\gamma-1)} \tilde{u}_1 \\ \eta_\infty^2 \tilde{u}_1 - \tilde{u}_{1yy} \\ \eta_\infty^2 \tilde{T}_1 - \frac{1}{\sigma^2} \tilde{T}_{1yy} \end{array} \right\} = \mathbf{0}, \qquad (3.27)$$

where $\sigma^2 = Pr$ and $\eta_\infty^2 = i(\omega - Mk)/\xi$ with $\mathrm{Re}(\eta_\infty) > 0$. This can now be solved analytically

$$\tilde{u}_1 = \tilde{u}_1^A e^{\eta_\infty y} + \tilde{u}_1^B e^{-\eta_\infty y},$$
$$\tilde{T}_1 = \tilde{T}_1^A e^{\sigma \eta_\infty y} + \tilde{T}_1^B e^{-\sigma \eta_\infty y},$$
$$\tilde{v}_1 = \tilde{v}_{1\infty} + \frac{ik}{\eta_\infty}(\tilde{u}_1^A e^{\eta_\infty y} - \tilde{u}_1^B e^{-\eta_\infty y}) + \frac{(\gamma-1)\eta_\infty \xi}{\sigma}(\tilde{T}_1^A e^{\sigma \eta_\infty y} - \tilde{T}_1^B e^{-\sigma \eta_\infty y}),$$

where $\tilde{u}_1^A$, $\tilde{u}_1^B$, $\tilde{T}_1^A$, $\tilde{T}_1^B$ and $\tilde{v}_{1\infty}$ are constants. Since $\mathrm{Re}(\eta_\infty) > 0$ the $\tilde{u}_1^A$ and $\tilde{T}_1^A$ terms grow exponentially as $y \to \infty$. In the outer region we have a Bessel function solution which does not grow exponentially away from the wall. So in order to match to the outer region only the decaying solutions can be allowed, i.e. we must set $\tilde{u}_1^A = \tilde{T}_1^A = 0$. This can be reformulated as the following boundary conditions at $y = Y$ which admit only the decaying solutions.

$$\tilde{u}_{1y} + \eta_\infty \tilde{u}_1 = 0, \quad \tilde{v}_1 = \tilde{v}_{1\infty} - \frac{\eta_\infty(\gamma-1)\xi}{\sigma}\tilde{T}_1 - \frac{ik}{\eta_\infty}\tilde{u}_1, \quad \tilde{T}_{1y} + \sigma\eta_\infty \tilde{T}_1 = 0 \quad \text{at } y = Y. \ (3.28)$$

Using these boundary conditions our inner solution can now be solved and the value $\tilde{v}_{1\infty}$ can be used to match to the outer region soluion. This gives the results of [4]. Note that the $\mathcal{O}(\varepsilon/\delta)$ amplification in the boundary layer does not propagate into the centre of the duct where the acoustics remain $\mathcal{O}(\varepsilon)$. This is because both $\tilde{u}_1$ and $\tilde{T}_1$ decay to zero outside the boundary layer, while $-\delta\tilde{v}_{1\infty}$, which is $\mathcal{O}(\varepsilon)$, is matched to the outer.

It should be noted that taking the decaying solution involves taking the correct branch of the square root of η_∞^2 so that $\mathrm{Re}(\eta_\infty) > 0$. This leads to a branch cut in the complex k-plane, with the branch point at $k = \omega/M$ and the branch cut extending vertically downwards towards
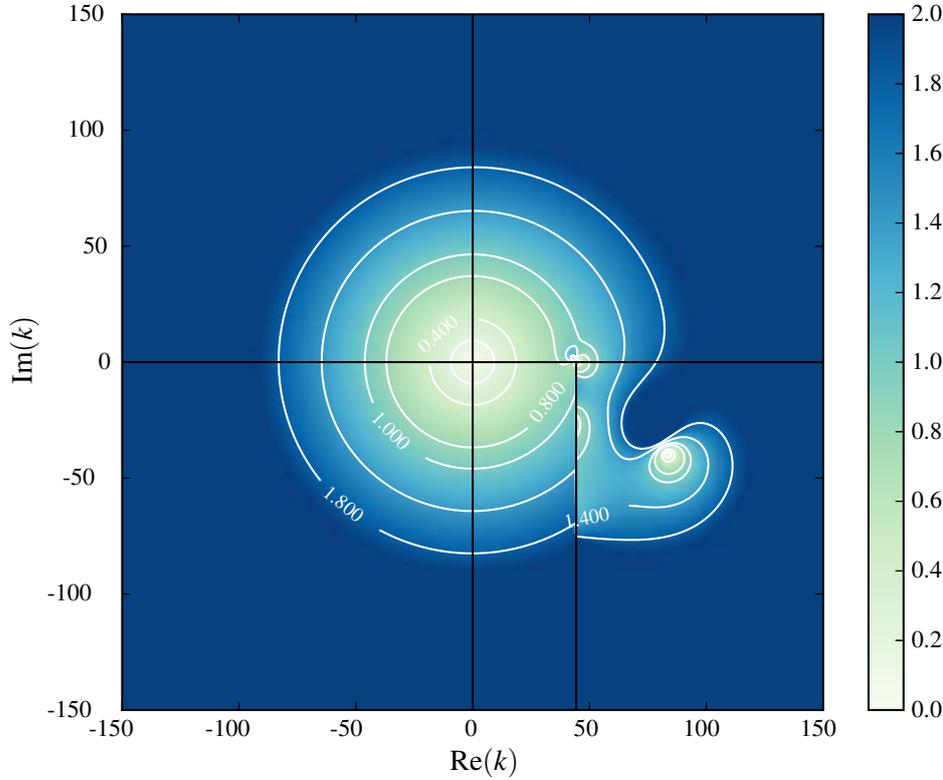
Fig. 3.1 Surface plot of $|Z_{\text{eff}}/Z_{\text{eff}}^{\text{Myers}} - 1|$ in the k-plane for $\omega = 31$, $M = 0.7$, $\delta = 10^{-3}$, $\text{Pr} = 0.7$ and $\xi = 1$. The lighter shades are where the Myers condition agrees fairly closely with the viscous asymptotics whereas the darker shades are where the two schemes disagree.

$-i\infty$. This branch cut can be seen prominently in figure 3.1, which plots $|Zeff/Z_{\text{eff}}^{\text{Myers}} - 1|$ in the k-plane for the leading order viscous asymptotics. Here, $Z_{\text{eff}} = \tilde{p}_O(1)/\tilde{v}_O(1)$ is the impedance seen by the outer solution at the wall, and therefore includes the effect of both the liner and the boundary layer. Figure 3.1 therefore compares this effective impedance with the impedance from the Myers boundary condition.

The same procedure as given here may be used to calculate the first order linear correction terms (quantities labelled '3' above), and such an analysis is given in [18]. Since these first order linear correction terms are not needed for calculating the nonlinear correction terms below, I will not reproduce this argument here.

Figure 3.2 shows the inner and outer solutions for the leading order linear acoustics compared to the solutions from the linearised Navier Stokes. It can be seen that the solutions
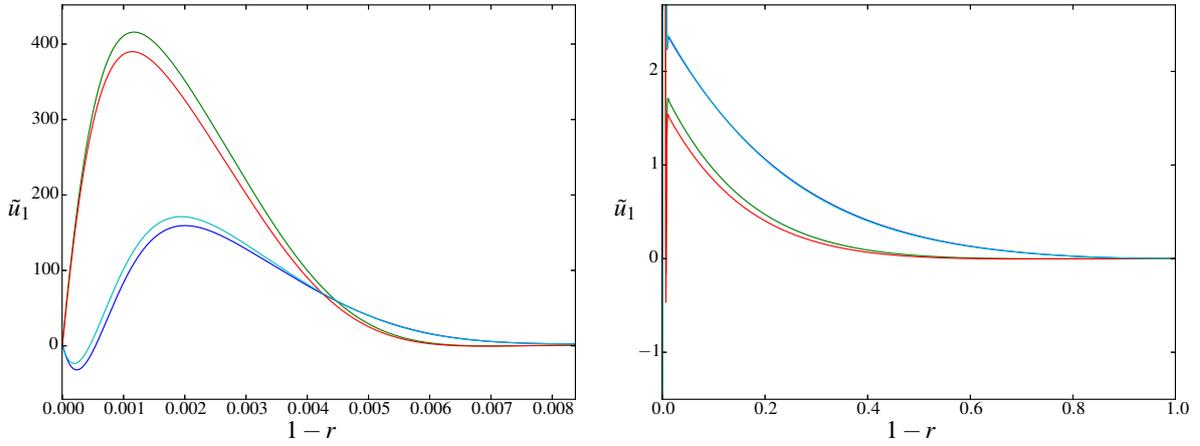
Fig. 3.2 Inner (left) and outer (right) mode shapes of the linear acoustic mode $\tilde{u}_1/\varepsilon$, comparing the asymptotics (real part dark blue, imaginary part green) to the linearised Navier Stokes (real part red, imaginary part red). Parameters are $M = 0.7$, $\delta = 10^{-3}$, $\mathrm{Pr} = 0.7$, $\xi = 0.8$, $\omega = 5$, $k = 5 + \mathrm{i}$ and $m = 2$

are in good agreement in both regions and the $\mathcal{O}(1/\delta)$ amplification inside the boundary layer is evident.

3.5 Nonlinear acoustics

We now turn our attention to the nonlinear correction terms (quantities labelled '2' in equation 3.3). Substituting the asymptotic ansatz (3.3) into the governing equations (2.1) and taking terms of order $\mathcal{O}(\varepsilon^2)$ will result in a set of linear ODEs to solve for the nonlinear correction terms, forced by terms quadratic in the leading order linear solution. Since these forcing terms are nonlinear, we must take the real parts of the perturbed leading order quantities before multiplying. For example, the multiple of $\tilde{u}_1 = \hat{u}_1 \exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}$

and $\mathrm{d}\tilde{v}_1/\mathrm{d}x = -\mathrm{i}k\hat{v}_1\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}$ is

$$
\begin{aligned}
&\mathrm{Re}\big(\hat{u}_1\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}\big)\,\mathrm{Re}\big(-\mathrm{i}k\hat{v}_1\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}\big)\\
&= \frac{1}{2}\big(\hat{u}_1\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx-\mathrm{i}m\theta} + \hat{u}_1^\star\mathrm{e}^{-\mathrm{i}\omega^\star t+\mathrm{i}k^\star x+\mathrm{i}m\theta}\big)\frac{1}{2}\big(-\mathrm{i}k\hat{v}_1\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx-\mathrm{i}m\theta} + \mathrm{i}k^\star\hat{v}_1^\star\mathrm{e}^{-\mathrm{i}\omega^\star t+\mathrm{i}k^\star x+\mathrm{i}m\theta}\big)\\
&= \Big(-\frac{\mathrm{i}k\hat{u}_1\hat{v}_1}{4}\mathrm{e}^{2\mathrm{i}(\omega t-kx-m\theta)} + \frac{\mathrm{i}k^\star\hat{u}_1^\star\hat{v}^\star}{4}\mathrm{e}^{-2\mathrm{i}(\omega^\star t+k^\star x+m\theta)}\Big)\\
&\quad + \Big(\frac{\mathrm{i}k^\star\hat{u}_1\hat{v}_1^\star}{4} - \frac{\mathrm{i}k\hat{u}_1^\star\hat{v}_1}{4}\Big)\mathrm{e}^{\mathrm{i}(\omega-\omega^\star)t-\mathrm{i}(k-k^\star)x}\\
&= \frac{1}{2}\mathrm{Re}\Big(-\mathrm{i}k\hat{u}_1\hat{v}_1\exp\{2\mathrm{i}\omega t - 2\mathrm{i}kx - 2\mathrm{i}m\theta\} + \mathrm{i}k^\star\hat{u}_1\hat{v}_1^\star\exp\{\mathrm{i}(\omega-\omega^\star)t - \mathrm{i}(k-k^\star)x\}\Big)\\
&= \frac{1}{2}\mathrm{Re}\big(-\mathrm{i}k\tilde{u}_1\tilde{v}_1 + \mathrm{i}k^\star\tilde{u}_1\tilde{v}_1^\star\big) = \frac{1}{4}\big(-\mathrm{i}k\tilde{u}_1\tilde{v}_1 + \mathrm{i}k^\star\tilde{u}_1\tilde{v}_1^\star\big) + c.c. \tag{3.29}
\end{aligned}
$$

where a '$\star$' denotes the complex conjugate. This therefore results in two different modes: a mode of double the frequency and wavenumber of the leading order acoustics ($\Omega = 2\omega$, $K = 2k$ and $\mathcal{M} = 2m$); and a 'zero' frequency mode that has the purely imaginary frequency ($\Omega = \omega - \omega^\star$) and wavenumber ($K = k - k^\star$ and $\mathcal{M} = 0$). It is useful to note that for the 'zero' frequency terms we are free to take the complex conjugate of the nonlinear terms while deriving the equations since they will have the same real part. However this is not possible for the double mode terms as the complex conjugate has a different frequency.

3.5.1 Weakly nonlinear equations

I will now derive the equations for the leading order weakly nonlinear acoustics. These occur due to the self interaction of a leading order mode and so will be a linear system of equations forced by terms that are quadratic in the leading order linear acoustic quantities. It is clear that the terms involving the leading order nonlinear '2' terms will have the same form as the equations for the linear acoustics, so we only need to calculate the quadratic forcing terms $\mathcal{Q}(\tilde{u}_1, \tilde{v}_1, \tilde{T}_1; \tilde{u}_1^\star, \tilde{v}_1^\star, \tilde{T}_1^\star; \omega, k)$. We begin as in §3.4.1 by considering the equation of state for a perfect gas (2.1e), now expanding to $\mathcal{O}(\varepsilon^2)$

$$
\rho + \varepsilon\tilde{\rho}_1 + \varepsilon^2\tilde{\rho}_2 = \frac{\gamma p}{(\gamma-1)T} + \frac{\varepsilon}{(\gamma-1)T}\Big(\tilde{p}_1\gamma - \frac{\tilde{T}_1}{T}\Big) + \frac{\varepsilon^2}{(\gamma-1)T}\Big(\tilde{p}_2\gamma - \frac{\tilde{T}_2}{T} - \frac{\tilde{p}_1\tilde{T}_1\gamma}{T} + \frac{\tilde{T}_1\tilde{T}_1}{T^2}\Big).
$$

Using the results for the steady mean flow and leading order acoustics (3.4) to cancel the mean flow and leading order linear terms we can match the $\mathcal{O}(\varepsilon^2)$ terms. We also need to

use the nonlinear multiplication rule (3.29) above which then gives

$$\tilde{\rho}_2 = \frac{\tilde{p}_2\gamma}{(\gamma-1)T} - \frac{\tilde{T}_2}{(\gamma-1)T^2} - \frac{\tilde{p}_1\tilde{T}_1^\star\gamma}{4(\gamma-1)T^2} + \frac{\tilde{T}_1\tilde{T}_1^\star}{4(\gamma-1)T^3}, \tag{3.30}$$

where the complex conjugate is taken in the case of the 'zero' frequency mode and is ignored for the double frequency mode. This equation is again independent of δ and so it will hold in both regions. Now I consider the continuity equation (2.1a), considering only the leading order nonlinear $\mathcal{O}(\varepsilon^2)$ time harmonic terms this becomes

$$\mathrm{i}(\Omega - uK)\tilde{\rho}_2 - \mathrm{i}K\rho\tilde{u}_2 + \frac{1}{r}\rho\tilde{v}_2 + (\rho\tilde{v}_2)_r - \frac{\mathrm{i}\mathcal{M}}{r}\rho\tilde{w}_2 = \frac{\mathrm{i}K}{4}\tilde{\rho}_1\tilde{u}_1^\star + \frac{\mathrm{i}\mathcal{M}}{4r}\tilde{\rho}_1\tilde{w}_1^\star - \frac{(r\tilde{\rho}_1\tilde{v}_1^\star)_r}{4r}. \tag{3.31}$$

Equation (3.30) can now be used to eliminate $\tilde{\rho}_2$. If we first consider the outer region we can also use the results from the linear outer region (3.26) to simplify this equation. In the outer region $\rho = 1$ and equation (3.4) together with the linear outer solution $\tilde{T}_1 = \tilde{p}_1$ gives $\tilde{\rho}_1 = \tilde{p}_1$. This means that the equation simplifies to

$$\mathrm{i}(\Omega - MK)\gamma\tilde{p}_2 - \frac{\mathrm{i}(\Omega - MK)}{T}\tilde{T}_2 - \mathrm{i}K\tilde{u}_2 + \frac{1}{r}\tilde{v}_2 + \tilde{v}_{2r} - \frac{\mathrm{i}\mathcal{M}}{r}\tilde{w}_2 =$$
$$\frac{\tilde{p}_1^\star}{4}\left(\mathrm{i}K\tilde{u}_1 + \frac{\mathrm{i}\mathcal{M}}{r}\tilde{w}_1 + \mathrm{i}(\Omega - MK)(\gamma-1)\tilde{T}_1 - \frac{\tilde{v}_1}{r} - \tilde{v}_{1r}\right) - \frac{\tilde{p}_{1r}^\star\tilde{v}_1}{4}. \tag{3.32}$$

In the inner region, taking into account the amplification (3.3) of $\tilde{T}_1$ and $\tilde{T}_2$ the equation for the nonlinear density perturbations simplifies to

$$\frac{\tilde{\rho}_2}{\rho} = -\frac{\tilde{T}_2}{T} + \frac{\tilde{T}_1\tilde{T}_1^\star}{4T^2}. \tag{3.33}$$

So we can now use this along with the amplification (3.3) of $\tilde{u}$, $\tilde{v}$ and $\tilde{T}$ and the rescaling (3.2) in the inner region to simplify the leading order $\mathcal{O}\left(\frac{\varepsilon^2}{\delta^2}\right)$ part of equation (3.31)

$$\mathrm{i}(\Omega - uK)\tilde{T}_2 + \mathrm{i}KT\tilde{u}_2 + T_y\tilde{v}_2 - T\tilde{v}_{2y} = \frac{T_y}{2T}(\tilde{T}_1\tilde{v}_1^\star) - \frac{1}{4}(\tilde{T}_1\tilde{v}_1^\star)_y + \frac{\tilde{T}_1\tilde{T}_1^\star}{4T}\mathrm{i}(\Omega - uK) + \frac{1}{4}\mathrm{i}K\tilde{u}_1\tilde{T}_1^\star. \tag{3.34}$$

Now we move on to consider the streamwise momentum equation (3.9). In the outer region the assumption of $\mathcal{O}(1)$ gradients means that all viscous terms are then $\mathcal{O}(\delta^2)$ which allows us to again neglect the viscous terms, it is then easy to write down the nonlinear equation

$$\mathrm{i}(\Omega - MK)\tilde{u}_2 - \mathrm{i}K\tilde{p}_2 = -\frac{\tilde{p}_1^\star}{4}(\mathrm{i}(\omega - Mk)\tilde{u}_1) + \frac{\mathrm{i}k}{4}\tilde{u}_1^\star\tilde{u}_1 - \frac{1}{4}\tilde{v}_1^\star\tilde{u}_{1r} + \frac{\mathrm{i}m}{4r}\tilde{w}_1^\star\tilde{u}_1. \tag{3.35}$$

In the inner region the leading order terms are $\mathcal{O}\left(\frac{\varepsilon^2}{\delta^2}\right)$ and gradients in r are $\mathcal{O}(1/\delta)$. This means that we again have that the only viscous terms that will appear at leading order are those with two r-derivatives. We can then write down the equation in the inner region, again taking into account the amplification (3.3) and the rescaling (3.2)

$$
\begin{aligned}
&\mathrm{i}(\Omega - uK)\tilde{u}_2 + \tilde{v}_2 u_y - \xi(\gamma-1)^2 T(T\tilde{u}_{2y} + \tilde{T}_2 u_y)_y = \\
&\frac{\mathrm{i}k}{4}\tilde{u}_1^\star\tilde{u}_1 - \frac{1}{4}\tilde{v}_1^\star\tilde{u}_{1y} + \frac{\tilde{T}_1^\star}{4T}[\mathrm{i}(\omega - uk)\tilde{u}_1 + \tilde{v}_1 u_y] + \frac{\xi(\gamma-1)^2 T}{4}(\tilde{T}_1^\star\tilde{u}_{1y})_y.
\end{aligned}
\tag{3.36}
$$

If we now consider radial momentum equation (3.13) in the outer region we can again ignore the viscous terms. If we then consider the remaining terms it is easy to see that the leading order $\mathcal{O}(\varepsilon^2)$ balance will give

$$
\mathrm{i}(\Omega - MK)\tilde{v}_2 + \tilde{p}_{2r} = -\frac{\tilde{p}_1^\star}{4}(\mathrm{i}(\omega - Mk)\tilde{v}_1) + \frac{\mathrm{i}k}{4}\tilde{u}_1^\star\tilde{v}_1 - \frac{1}{4}\tilde{v}_1^\star\tilde{v}_{1r} + \frac{\mathrm{i}m}{4r}\tilde{w}_1^\star\tilde{v}_1 + \frac{\tilde{w}_1^\star\tilde{w}_1}{4r}. \tag{3.37}
$$

Now in the inner boundary layer region given the amplification (3.3) we find that the radial gradient of the nonlinear pressure term is $\mathcal{O}\left(\frac{\varepsilon^2}{\delta^2}\right)$. Given the rescaling of all acoustic normal velocity terms (3.2) the pressure gradient term cannot be balanced and so the equation in the boundary layer is again simply

$$
\tilde{p}_{2y} = 0. \tag{3.38}
$$

If we now consider the θ-momentum equation (3.16), for the outer solution it is again clear that the viscous terms will not appear at leading order and the leading order nonlinear equation has a similar form to the streamwise and radial equations

$$
\mathrm{i}(\Omega - MK)\tilde{w}_2 - \frac{\mathrm{i}\mathcal{M}}{r}\tilde{p}_2 = -\frac{\tilde{p}_1^\star}{4}(\mathrm{i}(\omega - Mk)\tilde{w}_1) + \frac{\mathrm{i}k}{4}\tilde{u}_1^\star\tilde{w}_1 - \frac{1}{4}\tilde{v}_1^\star\tilde{w}_{1r} + \frac{\mathrm{i}m}{4r}\tilde{w}_1^\star\tilde{w}_1 - \frac{\tilde{v}_1^\star\tilde{w}_1}{4r},
\tag{3.39}
$$

while for the inner region the expansion is similar to that for the streamwise momentum equation. Again the only viscous terms that will appear at leading order are those with two r-derivatives. This means that the leading order $\mathcal{O}\left(\frac{\varepsilon^2}{\delta}\right)$ inner equation is

$$
\begin{aligned}
&\mathrm{i}(\Omega - uK)\tilde{w}_2 - \xi(\gamma-1)^2 T(T\tilde{w}_{2y})_y - \mathrm{i}\mathcal{M}\tilde{p}_2 = \\
&\frac{\mathrm{i}k}{4}\tilde{u}_1^\star\tilde{w}_1 - \frac{1}{4}\tilde{v}_1^\star\tilde{w}_{1y} + \frac{\tilde{T}_1^\star}{4T}[\mathrm{i}(\omega - uk)\tilde{w}_1] + \frac{\xi(\gamma-1)^2 T}{4}(\tilde{T}_1^\star\tilde{w}_{1y})_y.
\end{aligned}
\tag{3.40}
$$

Finally we consider the energy equation (3.19). We can again note that in the outer region all κ and μ terms are negligible and since $\tilde{T}_1 = \tilde{p}_1$ in the outer region, the equation then

simplifies to

$$i(\Omega - MK)\tilde{T}_2 - i(\Omega - MK)\tilde{p}_2 = -\frac{i(\omega - uk)}{4}\tilde{p}_1^{\star}\tilde{T}_1. \tag{3.41}$$

In the inner region given the amplification (3.3) all the pressure terms will be negligible and the only terms that appear from the right hand side are those with two radial derivatives. So we can simplify the equation to get

$$i(\Omega - uK)\tilde{T}_2 + \tilde{v}_2 T_y - \xi(\gamma-1)^2 T\left[\frac{1}{Pr}(\tilde{T}_2 T)_{yy} + \tilde{T}_2(u_y)^2 + 2Tu_y\tilde{u}_{2y}\right] = \frac{ik}{4}\tilde{u}_1^{\star}\tilde{T}_1$$

$$- \frac{\tilde{v}_1^{\star}\tilde{T}_{1y}}{4} + \frac{\tilde{T}_1^{\star}}{4T}[i(\omega - uk)\tilde{T}_1 + \tilde{v}_1 T_y] + \frac{\xi(\gamma-1)^2 T}{4}\left[\frac{(\tilde{T}_1^{\star}\tilde{T}_{1y})_y}{Pr} + T\tilde{u}_{1y}^{\star}\tilde{u}_{1y} + 2u_y\tilde{T}_1^{\star}\tilde{u}_{1y}\right]. \tag{3.42}$$

In the outer region we can now combine equations (3.32), (3.35), (3.37), (3.39) and (3.20) as before to derive an equation for the pressure perturbations. Given that the left hand sides of these equations have the same form as the linear equations we will again obtain a Bessel's equation, but now forced by quadratic terms in the leading order linear acoustics

$$\tilde{p}_{2rr} + \frac{\tilde{p}_{2r}}{r} + \tilde{p}_2\left((\Omega - MK)^2 - K^2 - \frac{\mathcal{M}^2}{r^2}\right) = \mathcal{Q}_O(\tilde{u}_1, \tilde{v}_1, \tilde{T}_1; \tilde{u}_1^{\star}, \tilde{v}_1^{\star}, \tilde{T}_1^{\star}; \omega, k). \tag{3.43}$$

The derivation of $\mathcal{Q}_O$ is given in §3.5.4.

Now in the inner region the equations again uncouple and we can use (3.34), (3.36) and (3.42) to derive a forced system of second order linear ODEs. Given that the left hand side of these equations has the same for as those for the linear acoustics, the equations are

$$\mathcal{L}(\tilde{u}_2, \tilde{v}_2, \tilde{T}_2; \Omega, K) = \mathcal{Q}(\tilde{u}_1, \tilde{v}_1, \tilde{T}_1; \tilde{u}_1^{\star}, \tilde{v}_1^{\star}, \tilde{T}_1^{\star}; \omega, k), \tag{3.44}$$

where $\mathcal{L}$ is as given in (3.23) and $\mathcal{Q}$ is given in §3.5.2. In the double frequency mode case, $\Omega = 2\omega$, $K = 2k$ and the $\star$ is ignored. In the 'zero' mode case, $\Omega = \omega - \omega^{\star}$, $K = k - k^{\star}$ and the $\star$ denotes the complex conjugate. From (3.38) the acoustic pressure perturbation is again constant within the inner region and so may be found by matching to the outer region.

3.5.2 Inner solution in the boundary layer

In the inner region we can use equations (3.34), (3.36) and (3.42) to write down the forcing term $\mathcal{Q}$,

$$
\mathcal{Q} = \left\{
\begin{array}{c}
\frac{T_y}{2T}(\tilde{T}_1 \tilde{v}_1^\star) - \frac{1}{4}(\tilde{T}_1 \tilde{v}_1^\star)_y + \frac{\tilde{T}_1 \tilde{T}_1^\star}{4T}\mathrm{i}(\Omega - uK) + \frac{1}{4}\mathrm{i}K\tilde{u}_1 \tilde{T}_1^\star \\[2mm]
\frac{\mathrm{i}k}{4}\tilde{u}_1^\star \tilde{u}_1 - \frac{1}{4}\tilde{v}_1^\star \tilde{u}_{1y} + \frac{\tilde{T}_1^\star}{4T}[\mathrm{i}(\omega - uk)\tilde{u}_1 + \tilde{v}_1 u_y] + \frac{\xi(\gamma-1)^2 T}{4}(\tilde{T}_1^\star \tilde{u}_{1y})_y \\[2mm]
\frac{\mathrm{i}k}{4}\tilde{u}_1^\star \tilde{T}_1 - \frac{1}{4}\tilde{v}_1^\star \tilde{T}_{1y} + \frac{\tilde{T}_1^\star[\mathrm{i}(\omega-uk)\tilde{T}_1 + \tilde{v}_1 T_y]}{4T} + \frac{\xi(\gamma-1)^2 T}{4}\left[\frac{(\tilde{T}_1^\star \tilde{T}_{1y})_y}{Pr} + (T\tilde{u}_{1y}^\star + 2u_y\tilde{T}_1^\star)\tilde{u}_{1y}\right]
\end{array}
\right\}.
$$
(3.45)

Similarly to the linear case, when the extrapolation outside the boundary layer at $y = Y \gg 1$ is carried out we get exponential terms $\propto \exp(\pm N_\infty y)$ where $N_\infty^2 = \mathrm{i}(\Omega - MK)/\xi$. The equations for large y are

$$
N_\infty^2 \xi \tilde{T}_2 - \frac{1}{(\gamma-1)}\tilde{v}_{2y} + \frac{\mathrm{i}K}{(\gamma-1)}\tilde{u}_2 = -\frac{1}{4}(\tilde{T}_1 \tilde{v}_1^\star)_y + \frac{\tilde{T}_1 \tilde{T}_1^\star}{4}\mathrm{i}(\Omega - MK)(\gamma-1) + \frac{1}{4}\mathrm{i}K\tilde{u}_1 \tilde{T}_1^\star,
$$

$$
N_\infty^2 \xi \tilde{u}_2 - \xi \tilde{u}_{2yy} = \frac{\mathrm{i}(\omega - Mk)(\gamma-1)}{4}(\tilde{u}_1 \tilde{T}_1^\star) + \frac{\mathrm{i}k}{4}\tilde{u}_1 \tilde{u}_1^\star - \frac{1}{4}(\tilde{v}_1 \tilde{u}_{1y}^\star) + \frac{\xi(\gamma-1)}{4}(\tilde{T}_1 \tilde{u}_{1y}^\star)_y,
$$

$$
N_\infty^2 \xi \tilde{T}_2 - \frac{\xi \tilde{T}_{2yy}}{\sigma^2} = \frac{\mathrm{i}k\tilde{u}_1^\star \tilde{T}_1}{4} - \frac{\tilde{v}_1 \tilde{T}_{1y}^\star}{4} + \frac{\mathrm{i}(\omega - Mk)(\gamma-1)\tilde{T}_1 \tilde{T}_1^\star}{4} + \frac{\xi(\gamma-1)(\tilde{T}_1^\star \tilde{T}_{1y})_y}{4\sigma^2} + \frac{\xi \tilde{u}_{1y}\tilde{u}_{1y}^\star}{4}.
$$

The equations for $\tilde{u}_2$ and $\tilde{T}_2$ have now decoupled and may be solved analytically. But we have already shown that for large y, $\tilde{u}_1$ and $\tilde{T}_1$ must decay exponentially, so we find the right hand side decays exponentially. This means that to be able to match to the outer we again need to take the decaying exponential solutions for $\tilde{u}_2$ and $\tilde{T}_2$ and so we obtain boundary conditions similar to the leading order linear case

$$
\tilde{u}_{2y} + N_\infty \tilde{u}_2 = 0, \quad \tilde{v}_2 = \tilde{v}_{2\infty} - \frac{N_\infty(\gamma-1)\xi}{\sigma}\tilde{T}_2 - \frac{\mathrm{i}K}{N_\infty}\tilde{u}_2, \quad \tilde{T}_{2y} + \sigma N_\infty \tilde{T}_2 = 0 \quad \text{at } y = Y,
$$
(3.46)

where we have ignored the forcing terms as they decay exponentially and so are expected to be negligible for $y \geq Y$.

The double frequency mode behaves similarly to the leading order acoustics. The branch cut for N_∞ is the same as for η_∞, and we can take the decaying solution and rewrite the equations to ensure only this solution is admitted. However for the 'zero' mode N_∞^2 is always real, and the resulting behaviour depends on the sign of N_∞^2. For downstream decaying modes, $N_\infty^2 < 0$, and both exponentials have purely imaginary argument and oscillate without decaying. In effect, this is because in this case the whole lower-half k-plane is mapped to the branch cut under the transformation $\Omega = \omega - \omega^\star$, $K = k - k^\star$. This means that the $\mathcal{O}(\varepsilon^2/\delta^2)$

'zero' frequency amplification will propagate out of the boundary layer and into the centre of the duct. For upstream decaying modes, $N_\infty^2 > 0$, and the decaying solution may be taken similarly to the leading order case.

3.5.3 Interaction of multiple modes

We might also consider the nonlinear effect due to two different frequency leading order modes interacting. We now take the leading order acoustics as a superposition of two waves,

$$\tilde{u}_1 = \text{Re}\left(\hat{u}_{1a}e^{\mathrm{i}(\omega_a t - k_a x - m_a \theta)}\right) + \text{Re}\left(\hat{u}_{1b}e^{\mathrm{i}(\omega_b t - k_b x - m_b \theta)}\right). \tag{3.47}$$

When we take quadratic terms of this form we will get the self interaction double frequency mode, $\frac{1}{4}(\tilde{u}_{1a}^2 + \tilde{u}_{1a}^{\star 2})$, and 'zero' frequency mode, $\frac{1}{4}(\tilde{u}_{1a}\tilde{u}_{1a}^\star + \tilde{u}_{1a}^\star \tilde{u}_{1a})$, for each of the leading order linear modes $\tilde{u}_{1a}$ and $\tilde{u}_{1b}$. However we will also get two cross-interaction terms, as

$$\begin{aligned}
\tilde{u}_1^2 = & [\frac{1}{2}(\tilde{u}_{1a} + \tilde{u}_{1a}^\star) + \frac{1}{2}(\tilde{u}_{1b} + \tilde{u}_{1b}^\star)]^2 \\
= & \frac{1}{4}\Big[(\tilde{u}_{1a}^2 + \tilde{u}_{1a}^{\star 2}) + (\tilde{u}_{1a}\tilde{u}_{1a}^\star + \tilde{u}_{1a}^\star \tilde{u}_{1a}) \\
& + (\tilde{u}_{1b}^2 + \tilde{u}_{1b}^{\star 2}) + (\tilde{u}_{1b}\tilde{u}_{1b}^\star + \tilde{u}_{1b}^\star \tilde{u}_{1b}) \\
& + 2(\tilde{u}_{1a}\tilde{u}_{1b} + \tilde{u}_{1a}^\star \tilde{u}_{1b}^\star) + 2(\tilde{u}_{1a}\tilde{u}_{1b}^\star + \tilde{u}_{1a}^\star \tilde{u}_{1b})\Big],
\end{aligned}$$

so we can see that we obtain two pairs of nonlinear self-interaction modes, as described above, as well as two cross-interaction modes. These cross-interactions modes will have the forms

$$\tilde{u}_{2+} = \text{Re}\left(\hat{u}_{2+}e^{\mathrm{i}[(\omega_a + \omega_b)t - (k_a + k_b)x - (m_a + m_b)\theta]}\right), \tag{3.48}$$

$$\tilde{u}_{2-} = \text{Re}\left(\hat{u}_{2-}e^{\mathrm{i}[(\omega_a - \omega_b^\star)t - (k_a - k_b^\star)x - (m_a - m_b)\theta]}\right). \tag{3.49}$$

Noting the factor of 2 from above we can see that the system of equations we now have to solve are

$$\begin{aligned}
\mathscr{L}(\tilde{u}_{2+}, \tilde{v}_{2+}, \tilde{T}_{2+}; \Omega, K) = & \mathscr{Q}(\tilde{u}_{1a}, \tilde{v}_{1a}, \tilde{T}_{1a}; \tilde{u}_{1b}, \tilde{v}_{1b}, \tilde{T}_{1b}; \omega_a, k_a) \\
& + \mathscr{Q}(\tilde{u}_{1b}, \tilde{v}_{1b}, \tilde{T}_{1b}; \tilde{u}_{1a}, \tilde{v}_{1a}, \tilde{T}_{1a}; \omega_b, k_b),
\end{aligned} \tag{3.50}$$

with $\mathscr{Q}$ from (3.45), $\Omega = \omega_a + \omega_b$ and $K = k_a + k_b$, and

$$
\begin{aligned}
\mathscr{L}(\tilde{u}_{2-}, \tilde{v}_{2-}, \tilde{T}_{2-}; \Omega, K) =& \mathscr{Q}(\tilde{u}_{1a}, \tilde{v}_{1a}, \tilde{T}_{1a}; \tilde{u}_{1b}^\star, \tilde{v}_{1b}^\star, \tilde{T}_{1b}^\star; \omega_a, k_a) \\
&+ \mathscr{Q}(\tilde{u}_{1b}^\star, \tilde{v}_{1b}^\star, \tilde{T}_{1b}^\star; \tilde{u}_{1a}, \tilde{v}_{1a}, \tilde{T}_{1a}; -\omega_b^\star, -k_b^\star),
\end{aligned}
\tag{3.51}
$$

with $\Omega = \omega_a - \omega_b^\star$ and $K = k_a - k_b^\star$.

The magnitude of the matching outer solution depends on N_∞^2 in the same way as the self-interaction modes. For N_∞^2 real and negative the outer solution is $\mathscr{O}(\varepsilon^2/\delta^2)$, and for all other values of N_∞^2 it is $\mathscr{O}(\varepsilon^2)$. However, for N_∞^2 to be real and negative a rather particular choice of ω_a, ω_b, k_a and k_b is needed, and the usual case will be of $\mathscr{O}(\varepsilon^2)$ and therefore the amplification is expected to remain contained within the boundary layer for most nonlinear wave interactions.

3.5.4   Outer solution in the duct interior

In the case $N_\infty^2 > 0$ (typically when the original mode decays in the upstream direction) the behaviour of the 'zero' mode outside the boundary layer is similar to the leading order mode, the amplification in the boundary layer decays and we can match the $\mathscr{O}(\frac{\varepsilon^2}{\delta})$ acoustic normal velocity to the outer solution.

To derive the equation for the outer pressure we use equations (3.35), (3.37) (3.39) and (3.41) to substitute all nonlinear terms for the nonlinear pressure in equation (3.32). The resulting equation will also be valid for the double frequency nonlinear mode. This gives the following forced Bessel's equation

$$
\begin{aligned}
\tilde{p}_{2rr} + \frac{1}{r}\tilde{p}_{2r} + A^2 \tilde{p}_2 - \frac{\mathcal{M}^2}{r^2}\tilde{p}_2 =& \frac{(\gamma-1)\xi^2\eta_\infty^2 N_\infty^2}{4}\tilde{p}_1^\star\tilde{T}_1 - \frac{(\gamma-1)\xi^2 N_\infty^4}{4}\tilde{p}_1^\star\tilde{T}_1 \\
&- \frac{iK\xi N_\infty^2}{4}\tilde{p}_1^\star\tilde{u}_1 + \frac{\xi N_\infty^2}{4}\left(\frac{\tilde{v}_1}{r} + \tilde{v}_{1r}\right)\tilde{p}_1^\star + \frac{\xi N_\infty^2}{4}\tilde{v}_1\tilde{p}_{1r}^\star - \frac{i\mathcal{M}\xi N_\infty^2}{4r}\tilde{w}_1\tilde{p}_1^\star \\
&+ \frac{iK}{4}\left[\xi\eta_\infty^2\tilde{p}_1^\star\tilde{u}_1 - ik\tilde{u}_1^\star\tilde{u}_1 + \tilde{v}_1^\star\tilde{u}_{1r} - \frac{im}{r}\tilde{w}_1^\star\tilde{u}_1\right] \\
&+ \frac{1}{4r}\left[-\xi\eta_\infty^2\tilde{p}_1^\star\tilde{v}_1 + ik\tilde{u}_1^\star\tilde{v}_1 - \tilde{v}_1^\star\tilde{v}_{1r} + \frac{im}{r}\tilde{w}_1^\star\tilde{v}_1 + \frac{1}{r}\tilde{w}_1^\star\tilde{w}_1\right] \\
&+ \frac{1}{4}\left[-\xi\eta_\infty^2\tilde{p}_1^\star\tilde{v}_1 + ik\tilde{u}_1^\star\tilde{v}_1 - \tilde{v}_1^\star\tilde{v}_{1r} + \frac{im}{r}\tilde{w}_1^\star\tilde{v}_1 + \frac{1}{r}\tilde{w}_1^\star\tilde{w}_1\right]_r \\
&+ \frac{i\mathcal{M}}{4r}\left[-p_1^\star\eta_\infty^2\xi\tilde{w}_1 + ik\tilde{u}_1^\star\tilde{w}_1 - \tilde{v}_1^\star\tilde{w}_{1r} + \frac{im}{r}\tilde{w}_1^\star\tilde{w}_1 - \frac{1}{r}\tilde{v}_1^\star\tilde{w}_1\right], \quad (3.52)
\end{aligned}
$$

where $A^2 = -\xi^2 N_\infty^4 - K^2 = [\Omega - MK]^2 - K^2$. This can be transformed to a forced Bessel's equation of order $\mathcal{M}$. Note that the right hand side forcing terms are all $\mathscr{O}(\varepsilon^2)$, so the leading

order outer nonlinear pressure term is mainly affected by the matching to $\tilde{v}_{2\infty}$. We could use this to solve for $\tilde{p}_2$ asymptotically however here I will solve these equations numerically.

3.5.5 Outer solution for self-interaction mode with $N_\infty^2 < 0$

In the case $N_\infty^2 < 0$, the extrapolation of the inner gives an oscillatory solution. This propagates into the rest of the duct and as the frequency of the oscillations is $\propto 1/\delta$, the gradients outside the boundary layer can no longer be assumed to be small and so the viscous terms cannot be ignored. However, an approximate solution to the outer equations may be found using the method of multiple scales. To do this we begin by defining variables y and R such that $r = R - \delta y$, so y is the rapidly varying variable and R the slowly varying variable. We can then expand all quantities as $\tilde{u}_2 = \frac{\varepsilon^2}{\delta^2}\tilde{u}_{O0} + \frac{\varepsilon^2}{\delta}\tilde{u}_{O1} + \varepsilon^2\tilde{u}_{O2}$ and expand the equations in powers of δ. This gives

$$\xi N_\infty^2 \tilde{u}_2 = \xi \delta^2 \tilde{u}_{2rr} + \frac{\xi \delta^2 \tilde{u}_{2rr}}{r} + \mathcal{O}(\varepsilon^2). \tag{3.53}$$

At leading order, $\mathcal{O}(\varepsilon^2/\delta^2)$, this simplifies to

$$N_\infty^2 \xi \tilde{u}_{O0} - \xi \tilde{u}_{O0yy} = 0 \qquad \Rightarrow \qquad \tilde{u}_{O0} = A_1(R)e^{ify} + A_2(R)e^{-ify}, \tag{3.54}$$

where $f^2 = -N_\infty^2$ so that f is real. At next order the equation is now

$$\tilde{u}_{O1yy} - N_\infty^2 \tilde{u}_{O1} = \left(2\tilde{u}_{O0Ry} + \frac{\tilde{u}_{O0y}}{R}\right) = 2ife^{ify}\left(A_1' + \frac{A_1}{2R}\right) - 2ife^{-ify}\left(A_2' + \frac{A_2}{2R}\right). \tag{3.55}$$

To avoid a secular term we require that both terms on the right hand side vanish, this gives the secularity conditions

$$A_1' + \frac{A_1}{2R} = 0 \quad \text{and} \quad A_2' + \frac{A_2}{2R} = 0 \qquad \Rightarrow \qquad \tilde{u}_{O0} = \frac{1}{\sqrt{R}}(A_1 e^{ify} + A_2 e^{-ify}). \tag{3.56}$$

Now this solution is singular at the origin, so to eliminate one of the two constants we need to solve for $\tilde{u}_O$ in an inner-outer region about $r = 0$ where the $1/r$ terms become large. To do this we set $r = \delta q$ and expand our equations to leading order in δ. This gives

$$N_\infty^2 \xi \tilde{u}_{O0} - \xi \tilde{u}_{O0qq} - \frac{\xi}{q}\tilde{u}_{O0q} = 0 \qquad \Rightarrow \qquad \tilde{u}_{O0} = AJ_0(fq). \tag{3.57}$$

For large q this solution can be approximated by the standard result

$$\tilde{u}_{O0} \approx A\sqrt{\frac{2}{f\pi q}}\cos\left(fq - \frac{\pi}{4}\right).$$ (3.58)

This now has to match with our outer solution, which means that the outer solution must be

$$\tilde{u}_{O0} = \frac{\tilde{u}_{2\infty}}{\sqrt{R}}\cos\left(-fy - \frac{\pi}{4}\right) \qquad \text{and} \qquad A = \sqrt{\frac{f\pi}{2\delta}}\tilde{u}_{2\infty}.$$ (3.59)

This means that

$$\tilde{u}_{O0} = \sqrt{\frac{f\pi}{2\delta}}\tilde{u}_{2\infty}J_0(fq) = \sqrt{\frac{f\pi}{2\delta}}\tilde{u}_{2\infty}J_0\left(\frac{fr}{\delta}\right)$$ (3.60)

is a uniformly valid asymptotic solution outside the boundary layer. We can use a similar method to find the leading order terms of $\tilde{T}$, $\tilde{v}$ and $\tilde{p}$

$$\tilde{T}_{O0} = \sqrt{\frac{f\sigma\pi}{2\delta}}\tilde{T}_{2\infty}J_0\left(\frac{f\sigma r}{\delta}\right) \approx \frac{\tilde{T}_{2\infty}}{\sqrt{R}}\cos\left(-f\sigma y - \frac{\pi}{4}\right),$$ (3.61a)

$$\tilde{v}_{O0} = \frac{\delta i(k - k^\star)\tilde{u}_{2\infty}}{f\sqrt{R}}\sin\left(-fy - \frac{\pi}{4}\right) - \frac{\delta f\xi(\gamma-1)\tilde{T}_{2\infty}}{\sigma\sqrt{R}}\sin\left(-f\sigma y - \frac{\pi}{4}\right) + \hat{v}_O(R),$$ (3.61b)

$$\tilde{p}_{O1} = \delta^2\xi^2 N_\infty^2(\gamma-1)\left(2 + \frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} - \frac{1}{Pr}\right)\tilde{T}_{O0} + \hat{p}_O(R).$$ (3.61c)

   To find the slowly varying terms of $\tilde{v}_O$ and $\tilde{p}_O$, we have to expand to order ε^2 and consider only the non-oscillating parts of the solutions, a full derivation is presented in appendix B.1. We find that $\hat{v}_O(R)$ and $\hat{p}_O(R)$ satisfy the same outer equations as $\tilde{v}_{2O}$ and $\tilde{p}_{2O}$ in the case $N_\infty^2 > 0$.

   We can find the constants $\tilde{u}_{2\infty}$ and $\tilde{T}_{2\infty}$ by matching to the inner solution at $y = Y \gg 1$. This gives

$$\tilde{u}_{2\infty} = \frac{1}{if}e^{ify + i\frac{\pi}{4}}\left(if\tilde{u}_2 - \tilde{u}_{2y}\right) \qquad \text{at } y = Y,$$ (3.62)

$$\tilde{T}_{2\infty} = \frac{1}{if\sigma}e^{if\sigma y + i\frac{\pi}{4}}\left(if\sigma\tilde{T}_2 - \tilde{T}_{2y}\right) \qquad \text{at } y = Y.$$ (3.63)

   Since the inner solutions are $\mathcal{O}(\varepsilon^2/\delta^2)$ and don't decay, the constants $\tilde{u}_{2\infty}$ and $\tilde{T}_{2\infty}$ are both $\mathcal{O}(\varepsilon^2/\delta^2)$. This corresponds to an amplified acoustic streaming, stronger than the $\mathcal{O}(\varepsilon^2)$

acoustic streaming that would be expected, that is caused by the viscous boundary layer over the acoustic lining.

3.6   Numerical results

While asymptotic approximate solutions to equations (3.23,3.45,3.50) are possible [see, e.g. 4, 17], here these equations are solved numerically using 4th order finite differences.

In the centre of the domain central differences are used while at the boundary 4th order forward differences are used to maintain accuracy near the boundary. It is important to use 4th order differences at the boundary as there can be fairly large gradients in the solution here particularly for the expanded full Navier Stokes solutions which need to be solved accurately. However it does come at the cost of having wide finite difference stencils near the boundary which increase the width of the resulting banded matrix.

The resulting $3N \times 3N$ banded matrix system of equations is solved using the LA-PACK_ZGBSV routine. To solve for the first order nonlinear inner, the same matrix is used, now forced by terms nonlinear in the leading order quantities. The system of equations is solved from $y = 0$ to $y = Y$, where Y is large enough so that the mean flow terms are approximately their uniform mean flow values, and the extrapolation condition (3.27) is used as the boundary condition.

By way of comparison, we also produce weakly nonlinear solutions to the full Navier Stokes equations, without any of the asymptotic assumptions in δ and the matching needed above. The full Navier Stokes equations are expanded in ε, full details of this derivation are given in appendix B.2, and a 4th order finite difference scheme is again used for the $\mathscr{O}(\varepsilon)$ and $\mathscr{O}(\varepsilon^2)$ equations thus obtained. In this case we get a $5N \times 5N$ banded matrix equation that is homogeneous in the leading order case and forced by leading order terms in the first order case. To accurately resolve the details in the boundary layer while still solving across the whole duct, stretched coordinates $\eta = \tanh(Sr)/\tanh(S)$ are used, where S is the stretching factor. This then concentrates the grid points about $r = 1$ so that the rapid variations there due to the thin boundary layer are properly resolved[1]. For the results below a stretching factor of $S = 2.0$ is used. While this is a fairly conservative stretching factor that requires many grid points to accurately resolve the detail in the boundary layer it avoids the problem of significant rounding errors that can occur near the boundary for larger stretching factors due to the very small step size right at the boundary. Before solving, the matrix is balanced

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[1]It should be noted that the use of this stretching factor is not necessary when solving for the asymptotic solutions as they are solved separately in the boundary layer and outer regions and so there is no problem of small scales that need to be resolved.
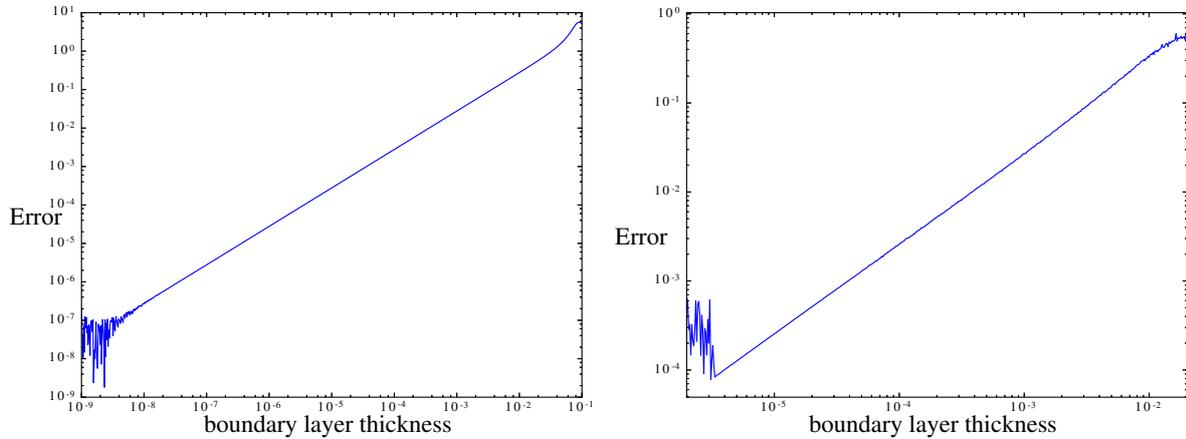
Fig. 3.3 Plot showing the convergence of the asymptotic linear (left) and nonlinear (right) solutions to the solutions from the linearised Navier Stokes and first order expansion in ε of the full Navier Stokes respectively as the boundary layer thickness $\delta \to 0$.

so that the largest value in each row is 1; this ensures that the solution remains stable near the origin, where terms involving $1/r$ can become large and could cause the matrix to be ill-conditioned.

The boundary conditions at the origin for the expansion in ε of the full Navier Stokes are found by assuming all quantities have a regular series expansion near the origin and matching powers of $1/r$, the full derivation is given in appendix B.3. This eliminates the possibility of any non-regular terms in the acoustic quantities and gives boundary conditions that are consistent with the expected Bessel function solutions. Figure 3.3 shows that the asymptotic linear and nonlinear solutions achieve the expected order of accuracy in δ, when compared to the expansion of the full Navier Stokes, as δ is decreased up to the point that rounding errors dominate the solutions.

Figure 3.4 shows plots of the mode shapes for both types of cross-interaction modes. The asymptotic solution can be seen to be in good agreement with the comparable result derived directly from the expansion in ε of the full Navier Stokes, calculated without assumptions about the asymptotics and matching. Note that these solutions are normalized so that $\tilde{p}_1 = 1$ at the wall. The values of ω and k used do not come from solving the dispersion relation but are used to show representative behaviours of the solutions.

A typical mode shape of the double-frequency nonlinear mode is given in figure 3.5. The nonlinear asymptotic solution is shown to be in good agreement with be first term from the expansion in ε of the full Navier Stokes, giving confidence in the asymptotic method applied. Moreover, both solutions are localized within the boundary layer ($\delta = 10^{-3}$ in this case), confirming the prediction that the $\mathscr{O}(1/\delta)$ amplification of the linear acoustics within the
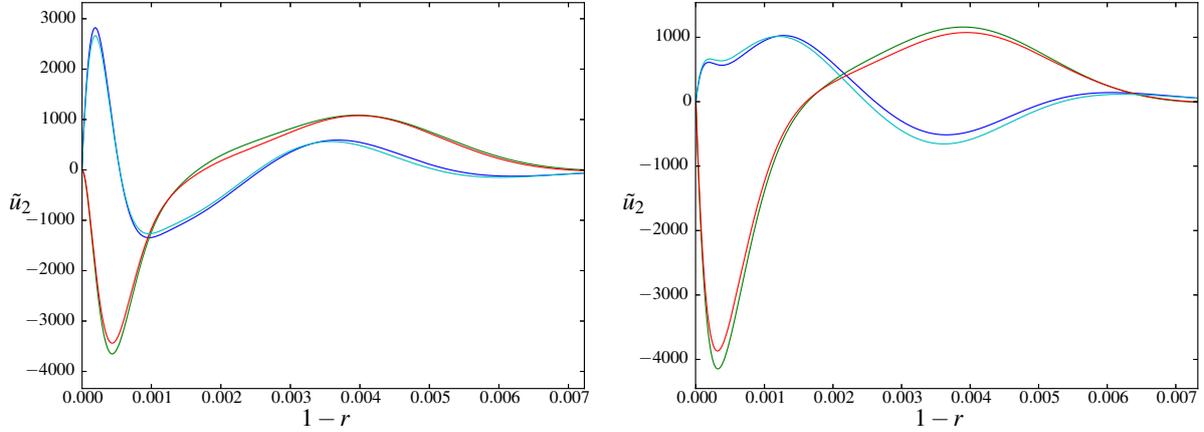
Fig. 3.4 Inner solutions of $\tilde{u}_{2+}/\varepsilon^2$ (left) and $\tilde{u}_{2-}/\varepsilon^2$ (right) for $\omega_a = 5$, $k_a = 10$, $m_a = 10$, $\omega_b = 31 + 5i$, $k_b = 12$, $m_b = 12$ for asymptotics (real part dark blue, imaginary part green) compared to expanded full Navier Stokes (real part light blue, imaginary part red), with $\delta = 10^{-3}$, $M = 0.7$, $\mathrm{Pr} = 0.7$ and $\xi = 0.8$
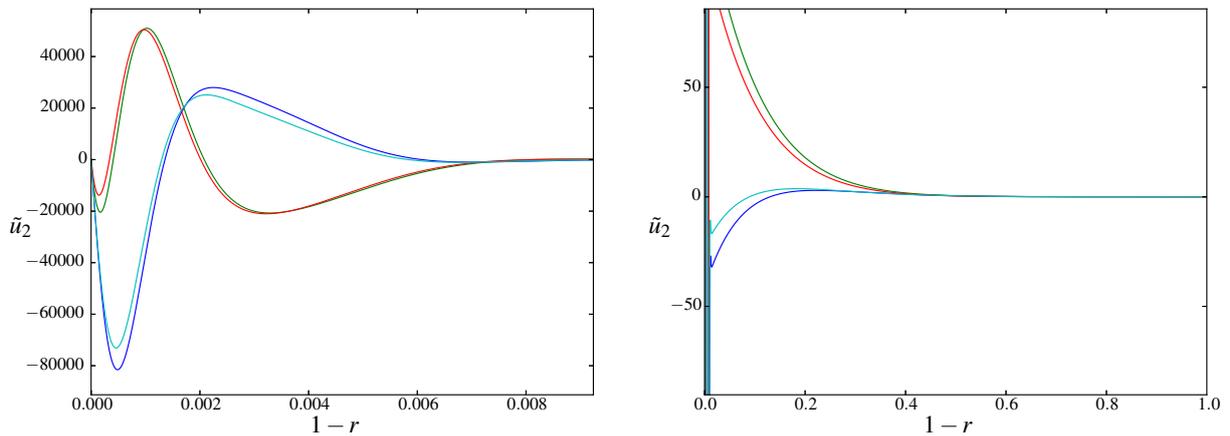
Fig. 3.5 Inner (left) and outer (right) mode shapes of the double frequency mode $\tilde{u}_2/\varepsilon^2$, comparing the asymptotics (real part dark blue, imaginary part green) to the first term from the expansion in ε of the full Navier Stokes (real part light blue, imaginary part red). Parameters are $M = 0.7$, $\delta = 10^{-3}$, $\mathrm{Pr} = 0.7$, $\xi = 0.8$, $\omega = 5$, $k = 5 + i$ and $m = 2$
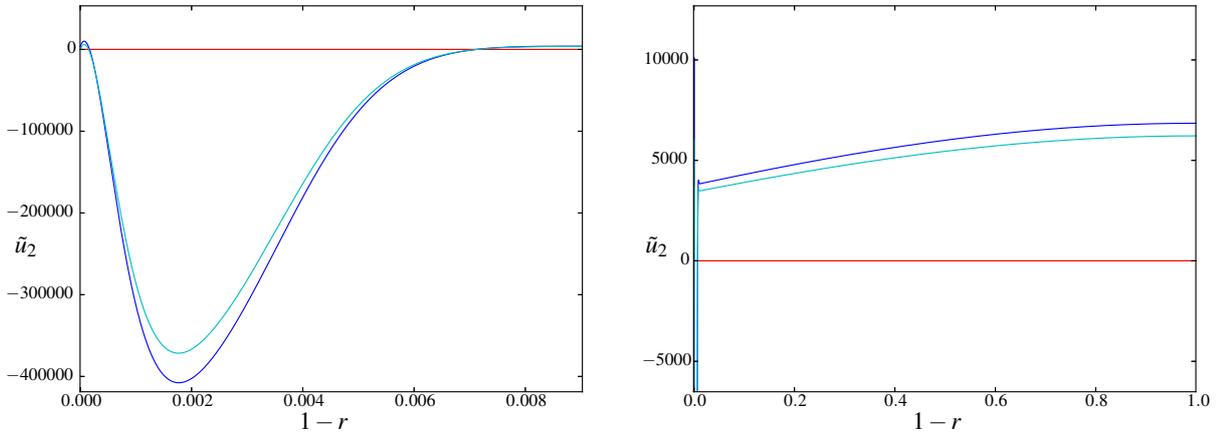
Fig. 3.6 Inner (left) and outer (right) of 'zero' mode $\tilde{u}_2/\varepsilon^2$ for $\omega = 5$, $k = 5 + i$ and $m = 2$ for asymptotics (real part dark blue, imaginary part green) compared to the first term from the expansion in ε of the full Navier Stokes (real part light blue, imaginary part red). Other parameters are $M = 0.7$, $\delta = 10^{-3}$, Pr $= 0.7$ and $\xi = 0.8$

boundary layer [4] does indeed trigger significantly more nonlinearity than would otherwise have been expected, but that, for the double frequency mode, it is mostly confined within the boundary layer.

The comparable 'zero' frequency nonlinear mode, for the case upstream decaying case $N_\infty^2 > 0$, is plotted in figure 3.6.[2] This shows a similar trend to 3.5, in that the predicted $\mathcal{O}(1/\delta^2)$ amplification of the nonlinear acoustics within the boundary layer is seen, but most of this does not bleed out into the rest of the duct; the acoustic streaming in the centre of the duct is $\mathcal{O}(\varepsilon^2/\delta)$.

In contrast, however, figure 3.7 shows the mode-shapes in the case of a downstream decaying mode, for which $N_\infty^2 < 0$. The solution is seen to oscillate rapidly in r with a radial wavelength of order $\mathcal{O}(\delta)$. This amplified rapid oscillation does not decay away from the boundary layer and is present throughout the duct, with an amplitude of $\mathcal{O}(\varepsilon^2/\delta^2)$. This shows that, in this case, the amplification within the boundary layer by a factor of $1/\delta$ previously predicted [4] does indeed lead to significant nonlinearity beyond what would have been expected within the duct, and that the nonlinearity is not confined to the boundary layer but bleeds out into the rest of the duct in this case.

Figure 3.8 shows the total sum of these effects, by plotting the overall perturbation to the streamwise velocity $\tilde{u}$ at different acoustic amplitudes. The effect of the nonlinear streaming

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[2]Note that for the 'zero' frequency mode I have combined both of the terms that contribute to the mode as it is time independent, this should be equivalent to taking the real part, and as can be seen from the plot the imaginary part of the solution is indeed zero.
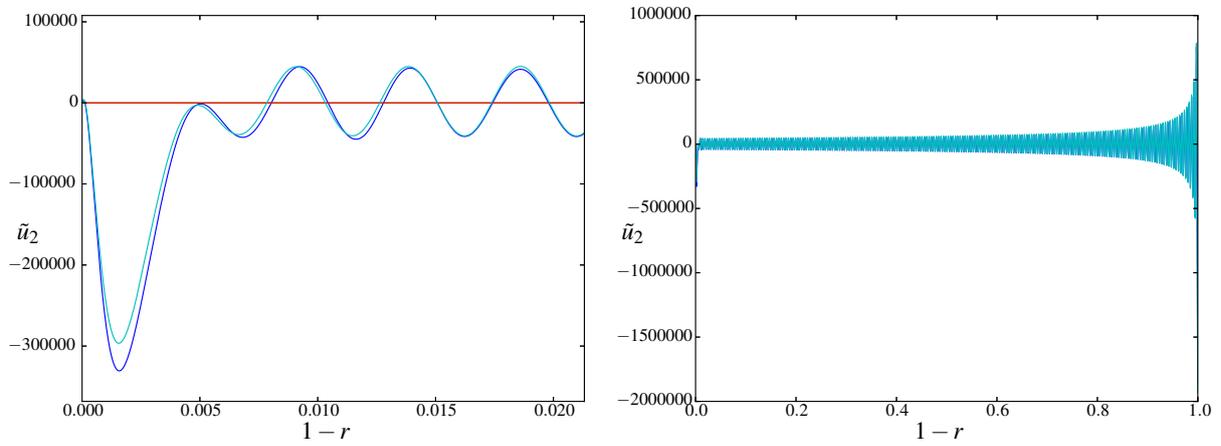
Fig. 3.7 Inner (left) and outer (right) of 'zero' mode $\tilde{u}_2/\varepsilon^2$ for $\omega = 5$, $k = 5 - \mathrm{i}$ and $m = 2$ for asymptotics (real part dark blue, imaginary part green) compared to the first term from the expansion in ε of the full Navier Stokes (real part light blue, imaginary part red). Other parameters are $M = 0.7$, $\delta = 10^{-3}$, $\mathrm{Pr} = 0.7$ and $\xi = 0.8$
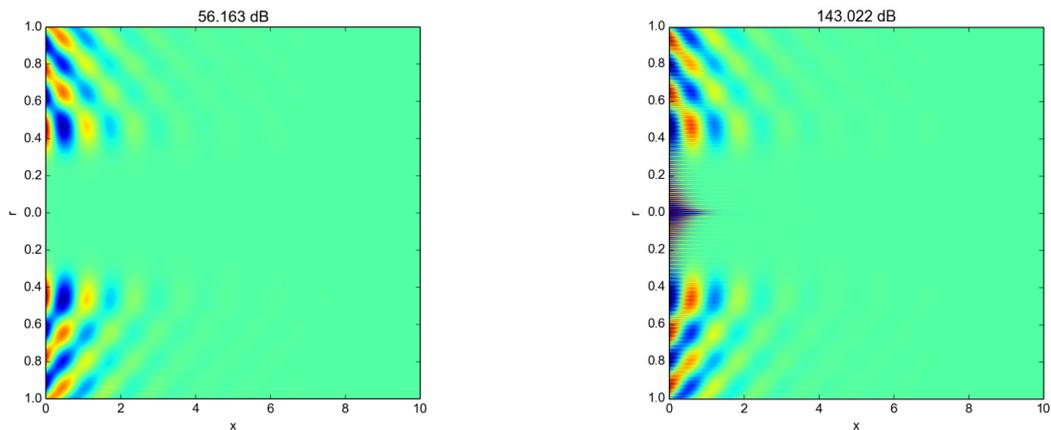
Fig. 3.8 Plot of snapshots of the total perturbation for $k = 5 - i$, $\omega = 31$, $m = 10$ and $\delta = 10^{-3}$ for different initial amplitudes.
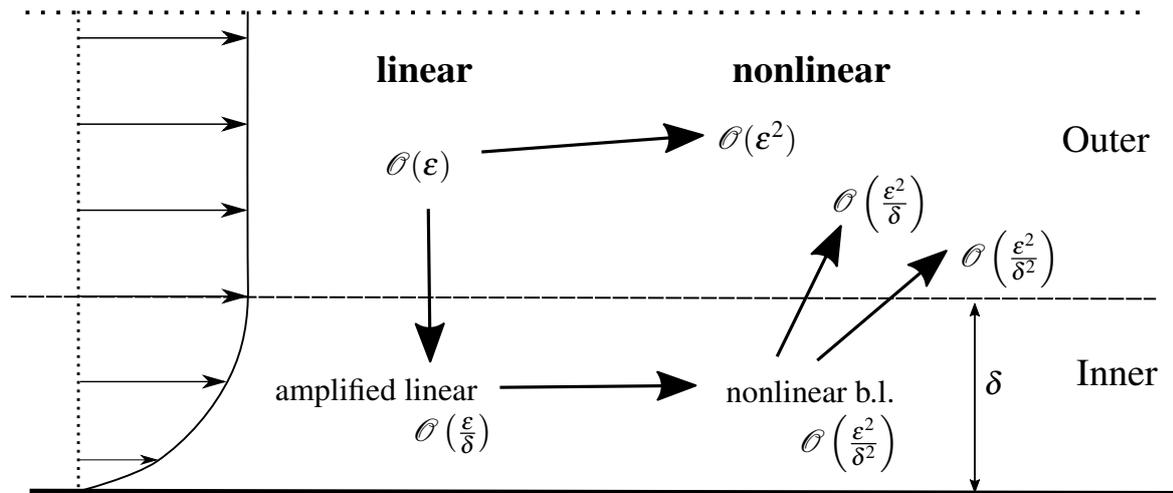
Fig. 3.9 Diagram showing the mechanism by which the linear amplification within the boundary layer causes an amplification of the outer nonlinear acoustics

is easily seen by the hairy appearance of the louder plot, although this nonlinear perturbation decays faster in the x direction than the damped acoustics, since the axial wavenumber has twice the decay rate of the linear acoustics.

3.7 Conclusion

In this chapter, I have investigated how the previously predicted [4] amplification of acoustics within a thin visco-thermal boundary layer over an acoustic lining leads to nonlinear effects becoming apparent at lower sound amplitudes than might have otherwise been predicted. It is emphasized that the nonlinearity presented here is nonlinearity within the fluid in the boundary layer, and is separate to the nonlinear behaviour of the actual boundary, such as the nonlinear behaviour of Helmholtz resonators near resonance [32].

As shown in figure 3.9, the mechanism is that sound of amplitude ε enters the boundary layer of thickness δ and is amplified to order ε/δ by the large shear. Nonlinear interactions inside the boundary layer then result in nonlinear acoustics with an amplitude of order ε^2/δ^2. These new acoustics have either double the frequency of the incoming sound, or 'zero' times the frequency, the latter corresponding to acoustic streaming. The double frequency amplified sound is mostly localized to the boundary layer, but it still causes amplified nonlinear acoustics of order ε^2/δ in the outer region. The upstream 'zero' frequency nonlinear modes behave in a similar fashion. However for downstream decaying sound the 'zero' frequency nonlinear modes bleed into the rest of the duct and show an ε^2/δ^2 amplitude throughout the duct. This is a factor of $1/\delta^2$ times greater than the magnitude of nonlinear

acoustics that would be created by ordinary nonlinear interactions outside of the boundary layer.

This is an unexpected result as one would not expect such highly oscillatory solutions in the centre of the duct. However this behaviour is present in both the asymptotic solution and in the results from the weakly nonlinear expansion of the Navier Stokes. It is possible that this behaviour is a consequence of one of our assumptions. In the next two chapters I will weaken some of these assumptions, in particular that of uniform mean flow, as in real aircraft engines we would expect to have a developing boundary layer profile. Also the effect of non-uniform mean flow has also been shown to be important for other duct acoustics problems [35].

Also derived here are equations governing the nonlinear interactions of two modes of differing frequencies. As for the self-interacting case, nonlinearity becomes important at lower amplitudes than expected due to the $1/\delta$ amplification within the boundary layer. Such interactions may well be important when two well-damped high azimuthal order spinning modes (for example, corresponding to the number of rotor and stator blades respectively) interact to produce a poorly-damped low azimuthal order nonlinear mode.

The analysis presented in this chapter assumed a thin boundary layer of constant width δ, and a small acoustic perturbation of amplitude ε, with $\varepsilon \ll \delta \ll 1$. In practice this will limit the maximum amplitude where the asymptotic solution may be used. Typically it is assumed that nonlinear effects are unimportant for sound power levels up to around 150dB, however here I have shown that the amplification within the boundary layer will cause nonlinear effects to become important much sooner than this. This means that the linear assumption in aircraft engines may only be valid up to around 100dB, much less than the typical sound level in an aircraft engine.

While the use of the asymptotics simplifies the governing equations, numerical solutions are still needed. In the linear case, other additional methods are used to derive approximate solutions and an effective impedance Z_{eff} that accounts for the behaviour within the boundary layer without having to numerically solve differential equations [4, 18, 17], and such techniques may well be applicable here.

Chapter 4

Non-parallel linear acoustics

4.1 Introduction

In the last chapter I used the common assumption that the background flow in the duct is parallel. This assumption is valid for large distances downstream and so may work well for experiments conducted on high aspect ratio apparatus (e.g. GFIT [14]). However, in this case we would no longer expect the boundary layer thickness to be very small and so $\mathcal{O}(\delta)$ corrections would be important; work on these correction terms for both inviscid and viscous background flows has been done previously [5, 17]. Indeed, in the limiting case that streamwise gradients are zero, we would expect the boundary layer thickness to be of the order of the radius of the duct, so the asymptotic expansion would break down entirely. However, we would only expect this to occur at very large downstream distances.

Even if we accept that the parallel flow assumption is valid for high aspect ratio apparatus, this assumption is less likely to be valid for aeroengines, whose aspect ratio is moderate (particularly since the diameter of turbofan engines has historically increased). Indeed, I showed in the previous chapter that under the parallel flow assumption, unusually large oscillatory acoustic streaming modes could appear. It is believed that these are unphysical and are an artefact of the parallel flow assumption.

In this chapter I will develop a non-parallel model for the linear acoustics inside a duct of slowly varying width, with a developing boundary layer shear background flow profile. This model will again take into account the effects of shear and viscosity, but now it will also be a non-local model that takes into account streamwise variations of the background flow. To do this I will use a three-layer matching WKB method to solve analytically for the acoustics in a viscous non-parallel boundary layer flow over an acoustic lining. In chapter §5 I will then extend this solution to consider the weakly nonlinear acoustics.
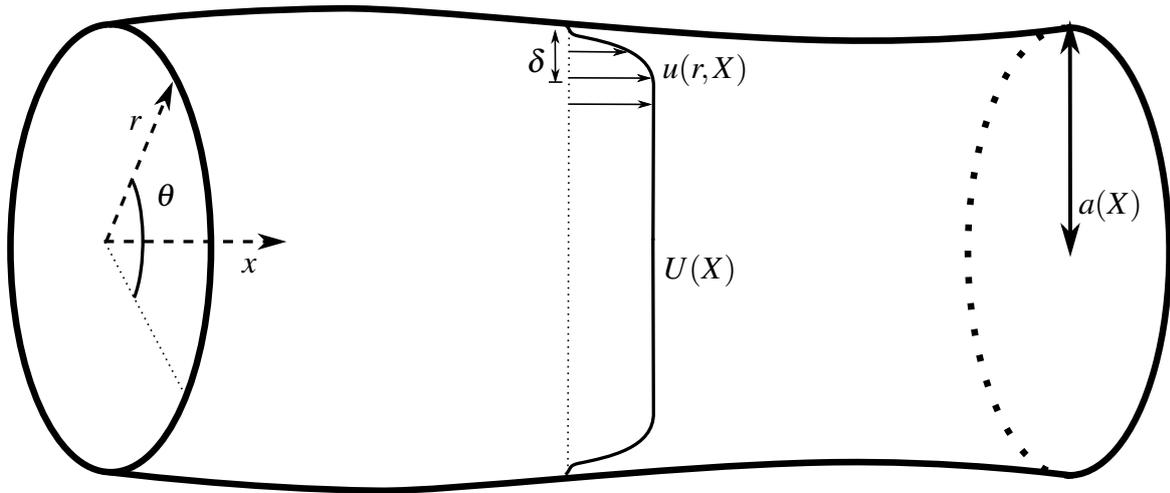
Fig. 4.1 Diagram of the slowly varying radius duct

To solve the problem I will introduce a new small variable $\varepsilon_k \ll 1$, which will be a measure of streamwise variation. It will quantify both the slow variation of the duct radius as well as the slow streamwise variation of the boundary layer mean flow. This means that I now have three small quantities ε, δ and ε_k. Since ε measures the amplitude of the acoustics, it is independent of the background flow. In this chapter I will only consider the linear acoustics and hence the leading order quantities in ε. In chapter §5 I will consider the weakly nonlinear acoustics which are the first order corrections in ε. The last small parameter, δ, is a measure of the boundary layer thickness. From the scaling for the acoustics (3.3) in chapter §3 we expect the leading order linear acoustics within the boundary layer to scale as $\mathscr{O}\left(\frac{\varepsilon}{\delta}\right)$. This requires that we have $\varepsilon \ll \delta$ so that it is still valid to assume that the acoustic quantities are much smaller than the mean flow quantities within the boundary layer. If this requirement did not hold the problem would become fully nonlinear within the boundary layer and would not be possible to solve analytically. As I do not wish to introduce further restrictions on the range of validity of the asymptotic solution, I will not assume any scaling between ε_k and δ and will solve for the first order corrections in both simultaneously. When we come to match between the inner and outer regions we will find that these corrections separate, as δ measures radial variations which are involved in the matching, while ε_k is a measure of streamwise variations which are solved using the WKB solution.

4.2   Statement of the problem

I consider the problem of acoustics in a cylindrical duct lined with an acoustic lining. We will allow the radius of the duct $a(X)$ to change slowly, depending on the slow variable $X = \varepsilon_k x$,

and assume that we have a developed boundary layer flow inside the duct. A diagram of the duct is shown in figure 4.1.

I again nondimensionalise as in §2.1, so that our lengths are nondimensionalised with respect to the duct radius l^* at $x = 0$ and use the mean flow sound speed in the centre of the duct at $x = 0$ to nondimensionalise our timescales. We still assume that the fluid inside the duct is a perfect gas and that the viscosity and thermal diffusivity are linear with temperature. The equations for our problem are then the Navier Stokes equations (2.1).

I assume that the acoustic lining reacts locally and can be modelled by a linear impedance boundary condition. In this chapter I will use the mass-spring-damper (1.3) impedance

$$Z = R + i\omega d - \frac{ib}{\omega},$$

however the method is valid for any local impedance which is a function of ω only.

4.2.1 Mean flow

For the outer mean flow I use the slowly varying solution from 2.3.2:

$$\pi a^2 = \frac{\pi M_0}{M} \left[\frac{1 + \frac{(\gamma-1)}{2}M^2}{1 + \frac{(\gamma-1)}{2}M_0^2} \right]^{\frac{(\gamma+1)}{2(\gamma-1)}}, \quad c_s^2 = \left[\frac{1 + \frac{(\gamma-1)}{2}M_0^2}{1 + \frac{(\gamma-1)}{2}M^2} \right],$$

$$\rho = c_s^{\frac{2}{(\gamma-1)}}, \quad P = \frac{1}{\gamma}\rho^\gamma, \quad T = \frac{\rho^{\gamma-1}}{\gamma-1}, \quad U = Mc_s, \quad V = \frac{\varepsilon_k a'(X)Ur}{a(X)},$$

since all outer mean flow terms depend algebraically on $a(X)$, they are also slowly varying downstream functions of X.

For the boundary layer flow profile I use the solutions to the boundary layer equations (2.23)

$$(\rho u)_x + (\rho v)_r = 0,$$
$$\rho(uu_x + vu_r) = -p_x + (\mu u_r)_r + \mu F_{rr}(x, r),$$
$$p_r = 0,$$
$$\rho(uT_x + vT_r) = up_x + vp_r + (\kappa T_r)_r + \mu u_r^2 + \mu G_{rr}(x, r),$$

for the results here I set both F and G to zero so that we can use the compressible Blasius boundary layer solution (2.31) and (2.38). Note that while in §2.4.2 I used the boundary layer coordinates $(\hat{x}, \hat{\zeta})$ (2.25), here in region II for simplicity I will use coordinates $(x, \zeta = \frac{a-r}{\delta\sqrt{x}})$.
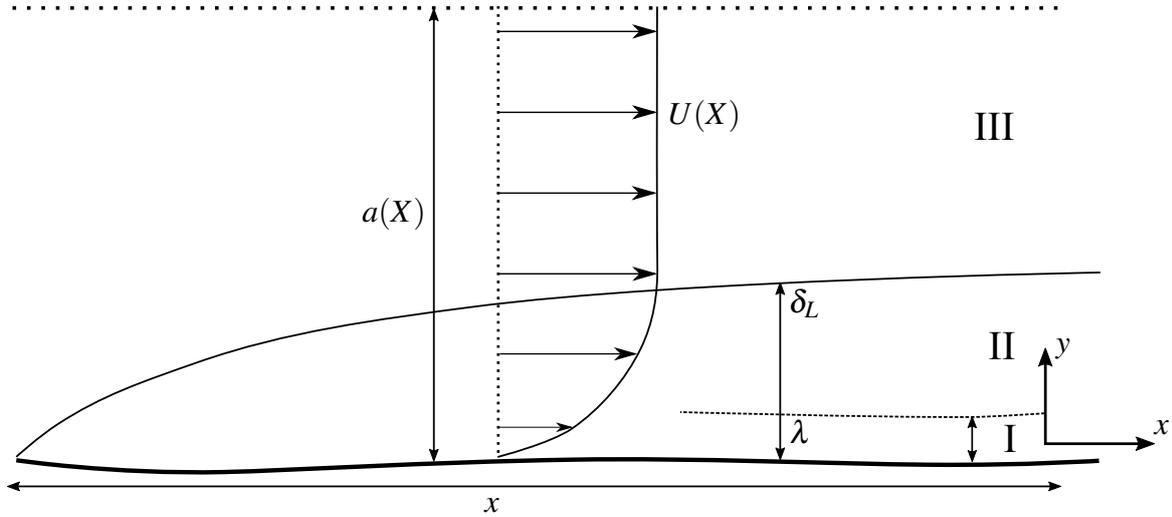
Fig. 4.2 Diagram of the duct with inner boundary layer region II and inner-inner acoustic boundary layer region I

Now that we have our background flow profile we can consider the asymptotic scalings of the problem to simplify the equations we need to solve.

4.3   Three layer setup-asymptotic model

From the standard Blasius boundary layer scaling (2.21) we have the thickness of the duct boundary layer $\delta_L = \sqrt{\frac{x}{Re}} = \delta\sqrt{x}$. Note that for simplicity, here we drop the M from the boundary layer thickness; this doesn't change the scalings since $M \sim \mathcal{O}(1)$. To attempt a closed form solution for the acoustics we consider the radial lengthscale λ over which the acoustics are affected by the viscosity. For the acoustics, the time dependence is given by the frequency $\mathrm{Re}(\omega) = \omega_r$ and the viscosity scales like $1/Re$, so the balance is now

$$\frac{\partial \tilde{u}}{\partial t} \sim \frac{\mu}{\rho}\tilde{u}_{yy} \quad \Longrightarrow \quad \omega_r \sim \frac{1}{Re\lambda^2},$$

rearranging we then have

$$\lambda \sim \delta_L\sqrt{\frac{1}{\omega_r x}} \sim \frac{\delta_L}{\sqrt{k_r x}}. \tag{4.1}$$

Note that the last result is true as we have $c_s \sim \mathcal{O}(1)$ when nondimensionalised and $\omega_r \sim c_s k_r$ where $k_r = \mathrm{Re}(k)$. Away from the leading edge we assume x is much larger than the wavelength, so that $k_r x \gg 1$. This lets us ignore any scattering effects due to the leading edge

of the duct. This also then implies that $\lambda \ll \delta_L$, so we can introduce an inner-inner region of lengthscale λ. We then have three distinct regions where different scalings dominate as shown in figure 4.2. In the inner-inner region, region I, the effect of viscosity on the acoustic dominates and the base flow can be viewed as approximately linear. Note that this region is sometimes knowns as the Stokes layer [30]. In region II, the effect of viscosity is negligible at leading order, so the acoustics act inviscidly, but the profile of the base flow must now be taken into account. In region III, the base flow is approximately uniform and the effect of viscosity is negligible at leading order, so the acoustics can be treated as being in an inviscid uniform flow, for which the solutions can be found to be in the form of Bessel functions.

Note we also need $x \gg \frac{\delta^2}{M}$ for boundary layer approximation to be appropriate. However this will be true away from a very small region near the leading edge.

If we compare these scalings to those used in §3 we can see that for the parallel assumption of §3 to hold we need to make x very large while keeping the mean flow boundary layer thickness δ_L constant. However if we keep δ_L constant as we increase x, our inner-inner region will become very thin. This means that it is not possible to take the limit of these new three layer scalings to obtain the same scalings as in §3 as doing so will collapse our inner-inner region into an infinitely thin region with infinitely large gradients near the wall of the duct.

4.4   Solution in each region

We first let $\varepsilon_k = \frac{1}{k_r x_0}$ where x_0 is some characteristic distance downstream, e.g. the location of a source and k_r is a characteristic wavenumber, perhaps related to the source frequency. This means that the effects of the slow variation of the radius of the duct and streamwise variation of the boundary layer will occur at the same order of magnitude and will be solved simultaneously for convenience. If we desired, we could then separate these effects afterwards by adjusting the magnitude of $a'(X)$.

Now since $1/k_r \ll x$ from §4.3, we have $\varepsilon_k \ll 1$, as required for the asymptotic scalings to hold. We then try a WKB solution of the form

$$\tilde{u} = (\tilde{u}_0(X,r) + \varepsilon_k \tilde{u}_1(X,r)) e^{-\mathrm{i}[\int^x \frac{k(X)}{\varepsilon_k} \,\mathrm{d}X - \omega t + m\theta]}, \tag{4.2}$$

where $X = \varepsilon_k x$ is a slow streamwise variable with which the background flow profile varies and x a fast streamwise variable with which the acoustics vary; $\tilde{u}_0$ is the leading order solution and $\tilde{u}_1$ the $\mathscr{O}(\varepsilon_k)$ correction term.

It is useful to note here that in chapter §3 I used $\tilde{u}_1$ as the leading order term and $\tilde{u}_3$ as a linear correction term. However here I use $\tilde{u}_0$ as the leading order term and $\tilde{u}_1$ and $\tilde{u}_3$ as linear correction terms.

4.4.1 Region III
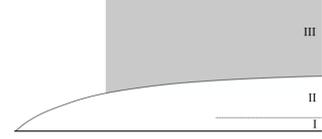
We first consider region III, the outer region. The radius of the duct is at $r = a(X)$ and we have assumed that the streamwise mean flow velocity, pressure and density are uniform but x-dependent, while the normal mean flow velocity is $\mathscr{O}(\varepsilon_k)$ and linear in the radial direction §2.3.2. We now consider the equations for the acoustic perturbations to first order in ε_k by expanding the linearised Navier Stokes equations. Since we assume that gradients are $\mathscr{O}(1)$ we can ignore the viscous terms in this region as they are $\mathscr{O}(\delta_L^2)$. The equations are then

$$\mathrm{i}(\omega - Uk)\frac{\tilde{\rho}}{\rho} - \mathrm{i}k\tilde{u} + \tilde{v}_r + \frac{\tilde{v}}{r} - \frac{\mathrm{i}m\tilde{w}}{r} + \varepsilon_k\left(\frac{U\tilde{\rho}_X}{\rho} + \frac{\tilde{u}\rho_X}{\rho} + \tilde{u}_X + \frac{\tilde{\rho}U_X}{\rho}\right) + \frac{(r\tilde{\rho}V)_r}{r\rho} = \mathscr{O}(\varepsilon_k^2),$$
(4.3a)

$$\rho\left[\mathrm{i}(\omega - Uk)\tilde{u} + \varepsilon_k(U\tilde{u}_X + \tilde{u}U_X) + V\tilde{u}_r\right] + \varepsilon_k\tilde{\rho}UU_X = \mathrm{i}k\tilde{p} - \varepsilon_k\tilde{p}_X + \mathscr{O}(\varepsilon_k^2),$$
(4.3b)

$$\rho\left[\mathrm{i}(\omega - Uk)\tilde{v} + \varepsilon_k U\tilde{v}_X + V\tilde{v}_r + \tilde{v}V_r\right] = -\tilde{p}_r + \mathscr{O}(\varepsilon_k^2),$$
(4.3c)

$$\rho\left[\mathrm{i}(\omega - Uk)\tilde{w} + \varepsilon_k U\tilde{w}_X + V\tilde{w}_r\right] = -\frac{1}{r}\rho V\tilde{w} + \frac{\mathrm{i}m}{r}\tilde{p} + \mathscr{O}(\varepsilon_k^2),$$
(4.3d)

$$\mathrm{i}(\omega - Uk)(\rho\tilde{T} - \tilde{p}) + \rho\left[\varepsilon_k(U\tilde{T}_X + \tilde{u}\mathcal{T}_X + \frac{\tilde{\rho}}{\rho}U\mathcal{T}_X) + V\tilde{T}_r\right] = \varepsilon_k(U\tilde{p}_X + \tilde{u}P_X) + V\tilde{p}_r + \mathscr{O}(\varepsilon_k^2).$$
(4.3e)

Since our outer mean flow solution $(\rho, U, V, \mathcal{T}, P)$ (2.18) is correct to order $\mathscr{O}(\varepsilon_k^2)$ there is no need to introduce correction terms due to the assumptions made when calculating the mean flow. We also consider the linearised the equation of state (2.1e) which gives an expression for $\tilde{\rho}$

$$\tilde{\rho} = \rho\left(\frac{\tilde{p}}{P} - \frac{\tilde{T}}{\mathcal{T}}\right).$$
(4.4)

At leading order these equations then simplify to

$$i(\omega - Uk)\left(\frac{\tilde{p}_0}{P} - \frac{\tilde{T}_0}{\mathcal{T}}\right) - ik\tilde{u}_0 + \tilde{v}_{0r} + \frac{\tilde{v}_0}{r} - \frac{im}{r}\tilde{w}_0 = 0,$$

$$\rho i(\omega - Uk)\tilde{u}_0 = ik\tilde{p}_0,$$

$$\rho i(\omega - Uk)\tilde{v}_0 = -\tilde{p}_{0r},$$

$$\rho i(\omega - Uk)\tilde{w}_0 = \frac{im}{r}\tilde{p}_0,$$

$$\rho i(\omega - Uk)\tilde{T}_0 = i(\omega - Uk)\tilde{p}_0.$$

These equations can now be combined to give a Bessel's equation for the acoustic pressure perturbation, similar to what we found in §3.4.1

$$\tilde{p}_{0rr} + \frac{\tilde{p}_{0r}}{r} + \tilde{p}_0\left(\frac{(\omega - Uk)^2}{(\gamma - 1)\mathcal{T}} - k^2 - \frac{m^2}{r^2}\right) = 0.$$

Due to the condition of regularity at the origin we find that the leading order solution for the acoustic perturbations in this region is

$$\tilde{p}_0 = B(X)J_m(\alpha r), \tag{4.5a}$$

$$\tilde{u}_0 = \frac{k}{\rho(\omega - Uk)}B(X)J_m(\alpha r), \tag{4.5b}$$

$$\tilde{v}_0 = \frac{-\alpha}{i\rho(\omega - Uk)}B(X)J_m'(\alpha r), \tag{4.5c}$$

$$\tilde{w}_0 = \frac{m}{r\rho(\omega - Uk)}B(X)J_m(\alpha r), \tag{4.5d}$$

$$\tilde{T}_0 = \frac{B(X)}{\rho}J_m(\alpha r), \tag{4.5e}$$

where $\alpha = \alpha(X) = \frac{(\omega - Uk)^2}{(\gamma - 1)\mathcal{T}} - k^2$ and $B(X)$ is an unknown slowly varying function that will be found by matching between each region and using a compatibility condition.

Now to find the order ε_k correction terms we first use equation (4.4) at order ε_k to cancel all $\tilde{\rho}$ terms in equations (4.3), which can then be written out

$$
\mathrm{i}(\omega - Uk)\left(\frac{\tilde{p}_1}{P} - \frac{\tilde{T}_1}{T}\right) - \mathrm{i}k\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 = R_p
$$
$$
= -\varepsilon_k\left(\frac{U}{\rho}\left(\frac{\tilde{p}_0}{(\gamma-1)T}\right)_X + \tilde{u}_0\frac{\rho_X}{\rho} + \tilde{u}_{0X} + U_X\frac{\tilde{p}_0}{\gamma P}\right) - 2\frac{V\tilde{p}_0}{\gamma Pr} - V\frac{\tilde{p}_{0r}}{\gamma P}, \tag{4.6a}
$$
$$
\rho\mathrm{i}(\omega - Uk)\tilde{u}_1 - \mathrm{i}k\tilde{p}_1 = R_u
$$
$$
= -\varepsilon_k\tilde{p}_{0X} - \varepsilon_k\left(UU_X\frac{\tilde{p}_0}{(\gamma-1)T} + \rho U\tilde{u}_{0X} + \rho\tilde{u}_0U_X\right) - \rho V\tilde{u}_{0r}, \tag{4.6b}
$$
$$
\rho\mathrm{i}(\omega - Uk)\tilde{v}_1 + \tilde{p}_{1r} = -\varepsilon_k(\rho U\tilde{v}_{0X} + \rho V\tilde{v}_{0r} + \rho\tilde{v}_0V_r) = R_v, \tag{4.6c}
$$
$$
\mathrm{i}(\omega - Uk)\tilde{w}_1 - \frac{\mathrm{i}m\tilde{p}_1}{\rho r} = -\frac{\rho V\tilde{w}_0}{r} - \rho V\tilde{w}_{0r} - \varepsilon_k\rho U\tilde{w}_{0X} = R_w, \tag{4.6d}
$$
$$
\mathrm{i}(\omega - Uk)(\rho\tilde{T}_1 - \tilde{p}_1) = \varepsilon_k U\tilde{p}_0\left(\frac{\rho_X}{\rho} - \frac{T_X}{(\gamma-1)T}\right) + \varepsilon_k\tilde{u}_0(P_X - \rho T_X) = R_T = 0, \tag{4.6e}
$$

where $R_T = 0$ by using the outer mean flow solution (2.18). These equations can now be combined to give a forced Bessel's equation for the acoustic pressure correction term

$$
\tilde{p}_{1rr} + \frac{\tilde{p}_{1r}}{r} + \tilde{p}_1\left(\alpha^2 - \frac{m^2}{r^2}\right) = -\rho\mathrm{i}(\omega - Uk)R_p - \mathrm{i}kR_u - \frac{\mathrm{i}m\rho R_w}{r} + (R_v)_r + \frac{R_v}{r}.
$$

Substituting the terms on the right hand side of this equation for the leading order solution, this becomes:

$$
\tilde{p}_{1rr} + \frac{\tilde{p}_{1r}}{r} + \tilde{p}_1\left(\alpha^2 - \frac{m^2}{r^2}\right)
$$
$$
= \frac{2\mathrm{i}V(\omega - Uk)}{(\gamma-1)T}\tilde{p}_{0r} + 2\mathrm{i}\left(k + \frac{(\omega - Uk)U}{(\gamma-1)T}\right)\varepsilon_k\tilde{p}_{0X}
$$
$$
- \mathrm{i}\varepsilon_k\tilde{p}_0\left(\frac{(\omega - Uk)UT_X}{(\gamma-1)T^2} - k_X - \frac{2k^2U_X}{\omega - Uk} - \frac{(\omega - Uk)U_X}{(\gamma-1)T} + \frac{U(\omega - Uk)\gamma\rho_X}{\rho(\gamma-1)T}\right.
$$
$$
\left. + \frac{(\gamma-1)k\rho_X}{\gamma\rho} + \frac{U^2k_X}{(\gamma-1)T} - \frac{kT_X}{\gamma T} - \frac{4V(\omega - Uk)}{\varepsilon_k r(\gamma-1)T} + \frac{2k^2V}{\varepsilon_k r(\omega - Uk)}\right),
$$

which can be simplified slightly using the outer mean flow solution (2.18) to give

$$
\tilde{p}_{1rr} + \frac{\tilde{p}_{1r}}{r} + \tilde{p}_1 \left(\alpha^2 - \frac{m^2}{r^2} \right) = 2\mathrm{i}k\varepsilon_k\tilde{p}_{0X} + \frac{2\mathrm{i}(\omega - Uk)}{(\gamma-1)\mathcal{T}} \left(V\tilde{p}_{0r} + U\varepsilon_k\tilde{p}_{0X} \right)
$$

$$
+ \varepsilon_k\mathrm{i}\tilde{p}_0 \left[k_X + \frac{(\omega-Uk)}{(\gamma-1)\mathcal{T}} \left(U_X + \frac{4V_r}{\varepsilon_k} \right) - \frac{k_XU^2}{(\gamma-1)\mathcal{T}} + 2k^2\frac{U_X - \frac{V_r}{\varepsilon_k}}{\omega-Uk} + \frac{(1-2\gamma)U\mathcal{T}_X(\omega-Uk)}{(\gamma-1)^2\mathcal{T}^2} \right]
$$

$$
\tag{4.7}
$$

4.4.2   Region II

In region II we transform into the boundary layer using the transformation $r = a(X) - \delta_L\zeta$, where ζ is our boundary layer mean flow parameter ($\zeta = y/\delta_L$, $\delta_L = \delta\sqrt{x}$). The derivatives then transform as follows

$$
\left.\frac{\partial}{\partial r}\right|_x \to -\frac{1}{\delta_L}\left.\frac{\partial}{\partial \zeta}\right|_X, \tag{4.8}
$$

$$
\left.\frac{\partial}{\partial x}\right|_r \to \left.\frac{\partial}{\partial x}\right|_\zeta + \varepsilon_k\left.\frac{\partial}{\partial X}\right|_\zeta + \varepsilon_k\left(\frac{a'(X)}{\delta_L} - \frac{\zeta}{2X} \right)\left.\frac{\partial}{\partial \zeta}\right|_X. \tag{4.9}
$$

Note that while in §2.4.2 we solved the boundary layer mean flow profile (ρ, u, v, T, p) in terms of $\hat{x}$ and $\hat{\zeta}$, the transformation to x and ζ does not change the asymptotic scaling of the coordinates. This is because the transformation integrals (2.25) do not change the scalings of $\hat{x}$ and $\hat{y}$ and, since $a_{\hat{x}} \sim \varepsilon_k$, the transformation $(X, \zeta) \to (\hat{x}, \hat{\zeta})$ also doesn't change the scalings, so the boundary layer flow profile gradients are $\mathcal{O}(1)$ in these coordinates. This can also be shown by considering the x and ζ derivatives using the chain rule, first for ζ,

$$
\left.\frac{\partial}{\partial \zeta}\right|_X = \delta\sqrt{x}\left.\frac{\partial}{\partial y}\right|_x = \delta\sqrt{x}c_s\rho\left.\frac{\partial}{\partial \hat{y}}\right|_{\hat{x}}
$$

$$
= \frac{\sqrt{x}Mc_s\rho}{\sqrt{\hat{x}}}\left.\frac{\partial}{\partial \hat{\zeta}}\right|_{\hat{x}}. \tag{4.10}
$$

Since the transformation from $\hat{x}$ to x doesn't change the scalings and the term $\frac{\sqrt{x}Mc_s\rho}{\sqrt{\hat{x}}}$ is $\mathcal{O}(1)$, the transformation from $\hat{\zeta}$ to ζ also won't change the scalings. Hence the ζ derivatives of

the mean flow boundary layer terms will be $\mathscr{O}(1)$. Now we consider X derivatives

$$
\begin{aligned}
\frac{\partial}{\partial X}\bigg|_\zeta &= \frac{1}{\varepsilon_k}\frac{\partial}{\partial x}\bigg|_y + \frac{y}{2X}\frac{\partial}{\partial y}\bigg|_x = \frac{\gamma p c_s}{\varepsilon_k}\frac{\partial}{\partial \hat{x}}\bigg|_{\hat{y}} + \left(\frac{\hat{y}_x}{\varepsilon_k} + \frac{y c_s \rho}{2X}\right)\frac{\partial}{\partial \hat{y}}\bigg|_{\hat{x}}, \\
&= \frac{\gamma p c_s}{\varepsilon_k}\frac{\partial}{\partial \hat{x}}\bigg|_{\hat{\zeta}} + \left(\frac{\sqrt{M}}{\delta\sqrt{\hat{x}}}\left(\frac{\hat{y}_x}{\varepsilon_k} + \frac{y c_s \rho}{2X}\right) - \frac{\gamma p c_s \hat{\zeta}}{2\hat{x}}\right)\frac{\partial}{\partial \hat{\zeta}}\bigg|_{\hat{x}}.
\end{aligned}
\tag{4.11}
$$

First note that when ζ is $\mathscr{O}(1)$ both y and $\hat{y}$ will be $\mathscr{O}(\delta_L)$. Also the mean flow boundary layer terms only depend on $\hat{x}$ through the outer mean flow terms that are functions of $a(x)$ and since $a_x \sim a_{\hat{x}} \sim \varepsilon_k$ all the coefficients above are $\mathscr{O}(1)$. So we also have that the X derivatives of all mean flow boundary layer terms are $\mathscr{O}(1)$.

In this region we will also use the scaling $v \to -v + \tilde{v} + \varepsilon_k a'(X)(u + \tilde{u})$ as in §2.4.2[1], since this causes the $\frac{a'(X)}{\delta_L}$ terms from the advective part of the equations to cancel. This is because the $\frac{a'(X)}{\delta_L}$ terms introduced by the change of the x-derivative (4.8) are cancelled out by the term added to v

$$
\begin{aligned}
uu_x + vu_r \to &\, u\left(u_x + \varepsilon_k(u_X + \frac{a'}{\delta_L} - \frac{\zeta}{2X})u_\zeta\right) - (v + \varepsilon_k a' u)\frac{u_\zeta}{\delta_L} \\
&= uu_x - v\frac{u_\zeta}{\delta_L} + \varepsilon_k\left(u_X - \frac{\zeta}{2X}u_\zeta\right).
\end{aligned}
$$

The mean flow term added to v, $\varepsilon_k a'(X)u$, is exactly the outer normal mean flow velocity V near the wall of the duct. This means that the remaining mean flow part v is only due to the boundary layer with $v \sim \mathscr{O}(\delta_L \varepsilon_k)$ when both X and ζ are $\mathscr{O}(1)$.

Using the same scaling arguments as in §3.3 we find that at leading order $\tilde{p}, \tilde{w}, \tilde{v} \sim \mathscr{O}(\varepsilon)$ and $\tilde{u}, \tilde{T} \sim \mathscr{O}(\frac{\varepsilon}{\delta_L})$. We then expand the equations to second order in δ_L and ε_k. Firstly the continuity equation (2.1a), including terms to order $\mathscr{O}(\varepsilon)$ and $\mathscr{O}(\frac{\varepsilon\varepsilon_k}{\delta_L})$ since the leading order terms are $\mathscr{O}(\frac{\varepsilon}{\delta_L})$,

$$
\begin{aligned}
i(\omega - uk)\tilde{\rho} + \varepsilon_k&\left(u\tilde{\rho}_X - \frac{\zeta u}{2X}\tilde{\rho}_\zeta + \tilde{u}\rho_X - \frac{\zeta\tilde{u}}{2X}\rho_\zeta\right) \\
-ik\rho\tilde{u} + \varepsilon_k&\left(\rho\tilde{u}_X - \frac{\zeta\rho}{2X}\tilde{u}_\zeta + \tilde{\rho}u_X - \frac{\zeta\tilde{\rho}}{2X}u_\zeta\right) \\
&- \frac{1}{\delta_L}\left(\rho\tilde{v}_\zeta - \tilde{\rho}v_\zeta - \tilde{\rho}_\zeta v + \rho_\zeta\tilde{v}\right) \\
+ \frac{1}{a(X)}(\rho\tilde{v} + \varepsilon_k a'(X)(\rho\tilde{u} + \tilde{\rho}u)) &- \frac{im\rho\tilde{w}}{a(X)} = \mathscr{O}(\delta_L^2, \varepsilon_k^2, \varepsilon_k\delta_L)\frac{\varepsilon}{\delta_L}.
\end{aligned}
$$

---

[1] Note that the $-$ sign in front of v here is a choice so that we remain consistent with the convention of §2.4.2

Now we consider the streamwise momentum equation (2.1b), which again has leading order terms that are $\mathcal{O}(\frac{\varepsilon}{\delta_L})$,

$$\rho\left[i(\omega-uk)\tilde{u}-\tilde{v}\frac{u_\zeta}{\delta_L}+\varepsilon_k\left(u\tilde{u}_X-\frac{u\zeta}{2X}\tilde{u}_\zeta+\tilde{u}u_X-\frac{\zeta\tilde{u}}{2X}u_\zeta\right)+\frac{v}{\delta_L}\tilde{u}_\zeta\right]$$

$$+\tilde{\rho}(\varepsilon_k(uu_X-\frac{u\zeta}{2X}u_\zeta)+\frac{v}{\delta_L}u_\zeta)$$

$$-ik\tilde{p}-\frac{a'(X)\varepsilon_k}{\delta_L}\tilde{p}_\zeta-\frac{(\gamma-1)\varepsilon_k}{X}(T\tilde{u}_\zeta+\tilde{T}u_\zeta)_\zeta=\mathcal{O}(\delta_L^2,\varepsilon_k^2,\varepsilon_k\delta_L)\frac{\varepsilon}{\delta_L}.$$

For the radial momentum equation only the pressure gradient term is $\mathcal{O}(\frac{\varepsilon}{\delta_L})$ while all others are of higher order,

$$\rho\left[i(\omega-uk)\tilde{v}+\varepsilon_k a'(X)i(\omega-uk)\tilde{u}-\tilde{v}\frac{\varepsilon_k a'(X)}{\delta_L}u_\zeta\right]-\frac{\tilde{p}_\zeta}{\delta_L}=\mathcal{O}(\delta_L^2,\varepsilon_k^2,\varepsilon_k\delta_L)\frac{\varepsilon}{\delta_L}. \quad (4.12)$$

The azimuthal momentum equation only has terms of order $\mathcal{O}(\varepsilon)$, so we only need to expand to leading order,

$$\rho\left[i(\omega-uk)\tilde{w}\right]-\frac{im}{a(X)}\tilde{p}=\mathcal{O}(\delta_L,\varepsilon_k)\varepsilon.$$

The energy equation (2.1d) again has leading order terms that are $\mathcal{O}(\frac{\varepsilon}{\delta_L})$,

$$\rho\left[i(\omega-uk)\tilde{T}-\tilde{v}\frac{T_\zeta}{\delta_L}+\varepsilon_k\left(u\tilde{T}_X-\frac{u\zeta}{2X}\tilde{T}_\zeta+\tilde{u}T_X-\frac{\zeta\tilde{u}}{2X}T_\zeta\right)+\frac{v}{\delta_L}\tilde{T}_\zeta\right]$$

$$+\tilde{\rho}(\varepsilon_k(uT_X-\frac{u\zeta}{2X}T_\zeta)+\frac{v}{\delta_L}T_\zeta)-i(\omega-uk)\tilde{p}-\varepsilon_k(\tilde{u}p_X-\frac{\zeta\tilde{u}}{2X}p_\zeta)-\tilde{v}\frac{p_\zeta}{\delta_L}$$

$$-\frac{\varepsilon_k(\gamma-1)}{X}\left[\frac{1}{Pr}(\tilde{T}T)_{\zeta\zeta}+\tilde{T}(u_\zeta)^2+2Tu_\zeta\tilde{u}_\zeta\right]=\mathcal{O}(\delta_L^2,\varepsilon_k^2,\varepsilon_k\delta_L)\frac{\varepsilon}{\delta_L}.$$

We can see in the equations above that the viscous terms only appear as an order ε_k correction, since $\mu\sim\delta^2\sim\frac{\varepsilon_k\delta_L^2}{X}$, and so the leading order solutions are only effected by the mean flow boundary layer profile shear. It is the interaction between the large streamwise boundary layer shear and the non-zero acoustic normal velocity perturbation $\tilde{v}$ that causes the $\mathcal{O}\left(\frac{\varepsilon}{\delta_L}\right)$ amplification in this region.

We now use the equation of state (2.1e) to write down the density perturbation in terms of the pressure and temperature perturbations. But since the pressure perturbation $\tilde{p}_0$ is $\mathcal{O}(\varepsilon)$

the density perturbation only depends on the temperature perturbation to leading order

$$\frac{\tilde{\rho}_0}{\rho} = -\frac{\tilde{T}_0}{T}.$$
(4.13)

Now $p_\zeta = \mathcal{O}(\delta_L^2 \varepsilon_k)$, so using (4.13) we can solve for the leading order acoustics, which gives

$$\tilde{p}_{0\zeta} = 0,$$
(4.14a)

$$\tilde{u}_0 = C(X)\frac{u_\zeta}{\delta_L},$$
(4.14b)

$$\tilde{v}_0 = C(X)\mathrm{i}(\omega - uk),$$
(4.14c)

$$\tilde{w}_0 = \tilde{p}_0 \frac{m}{a(X)\rho(\omega - uk)},$$
(4.14d)

$$\tilde{T}_0 = C(X)\frac{T_\zeta}{\delta_L},$$
(4.14e)

where we will find the slow functions of X, $\tilde{p}_0$ and $C(X)$, by matching to region III. Note that in §3.3 we scaled the leading order terms $\tilde{u}_1$ and $\tilde{T}_1$ by $\frac{1}{\delta}$ while here we include the $\frac{1}{\delta_L}$ term in the definition of $\tilde{u}_0$ and $\tilde{T}_0$, as δ_L varies with X.

Now for the first order terms we will have both $\mathcal{O}(\varepsilon_k)$ corrections and $\mathcal{O}(\delta_L)$ corrections. Here we will consider both corrections simultaneously. Doing this we will find equations for $\tilde{u}_1$, $\tilde{v}_1$ and $\tilde{T}_1$ forced by the leading order terms. We begin by considering the $\mathcal{O}\left(\frac{\varepsilon\varepsilon_k}{\delta_L}\right)$ and $\mathcal{O}(\varepsilon)$ terms in the continuity equation (2.1a)

$$\mathrm{i}(\omega - uk)\frac{\tilde{\rho}_1}{\rho} - \mathrm{i}k\tilde{u}_1 - \frac{\tilde{v}_{1\zeta}}{\delta_L} - \frac{\rho_\zeta \tilde{v}_1}{\rho\delta_L} = -\frac{\varepsilon_k}{\rho}\left(u\tilde{\rho}_{0X} - \frac{\zeta u}{2X}\tilde{\rho}_{0\zeta} + \tilde{u}_0\rho_X - \frac{\zeta\tilde{u}_0}{2X}\rho_\zeta\right)$$
$$-\frac{\varepsilon_k}{\rho}\left(\rho\tilde{u}_{0X} - \frac{\zeta\rho}{2X}\tilde{u}_{0\zeta} + \tilde{\rho}_0 u_X - \frac{\zeta\tilde{\rho}_0}{2X}u_\zeta\right) - \frac{1}{\delta_L\rho}\left(\tilde{\rho}_0 v_\zeta + \tilde{\rho}_{0\zeta}v\right)$$
$$-\frac{1}{a(X)}(\tilde{v}_0 + \varepsilon_k a'(X)(\tilde{u}_0 + \frac{\tilde{\rho}_0}{\rho}u)) + \frac{\mathrm{i}m\tilde{w}_0}{a(X)}.$$

We also consider the correction terms to the streamwise momentum equation (2.1b)

$$\mathrm{i}(\omega - uk)\tilde{u}_1 = \tilde{v}_1\frac{u_\zeta}{\delta_L} - \varepsilon_k\left(u\tilde{u}_{0X} - \frac{u\zeta}{2X}\tilde{u}_{0\zeta} + \tilde{u}_0 u_X - \frac{\zeta\tilde{u}_0}{2X}u_\zeta\right) - \frac{v}{\delta_L}\tilde{u}_{0\zeta}$$
$$-\frac{\tilde{\rho}_0}{\rho}(\varepsilon_k(uu_X - \frac{u\zeta}{2X}u_\zeta) + \frac{v}{\delta_L}u_\zeta) + \frac{\mathrm{i}k\tilde{p}_0}{\rho} + \frac{a'(X)\varepsilon_k}{\delta_L\rho}\tilde{p}_{0\zeta} + \frac{(\gamma-1)\varepsilon_k}{X\rho}(T\tilde{u}_{0\zeta} + \tilde{T}_0 u_\zeta)_\zeta,$$

and finally we consider the correction term to the energy equation (2.1d):

$$\mathrm{i}(\omega - uk)\tilde{T}_1 = \tilde{v}_1 \frac{T_\zeta}{\delta_L} - \varepsilon_k \left(u\tilde{T}_{0X} - \frac{u\zeta}{2X}\tilde{T}_{0\zeta} + \tilde{u}_0 T_X - \frac{\zeta \tilde{u}_0}{2X} T_\zeta \right) - \frac{v}{\delta_L}\tilde{T}_{0\zeta}$$
$$- \frac{\tilde{\rho}_0}{\rho}\left(\varepsilon_k (uT_X - \frac{u\zeta}{2X} T_\zeta) + \frac{v}{\delta_L} T_\zeta \right) + \frac{\mathrm{i}(\omega - uk)}{\rho}\tilde{p}_0 + \varepsilon_k \tilde{u}_0 \frac{p_X}{\rho}$$
$$+ \frac{\varepsilon_k(\gamma - 1)}{\rho X}\left[\frac{1}{Pr}(\tilde{T}_0 T)_{\zeta\zeta} + \tilde{T}_0(u_\zeta)^2 + 2Tu_\zeta \tilde{u}_{0\zeta} \right].$$

We now substitute the leading order results (4.14) into the right hand side of these equations. This gives the continuity equation

$$\mathrm{i}(\omega - uk)\frac{\tilde{\rho}_1}{\rho} - \mathrm{i}k\tilde{u}_1 - \frac{\tilde{v}_{1\zeta}}{\delta_L} - \frac{\rho_\zeta \tilde{v}_1}{\rho \delta_L} = -\frac{\varepsilon_k}{\rho}\left[u\left(\frac{C\rho_\zeta}{\delta_L}\right)_X + \rho\left(\frac{Cu_\zeta}{\delta_L}\right)_X \right]$$
$$-\frac{\varepsilon_k C}{\rho \delta_L}\left(-\frac{\zeta u}{2X}\rho_{\zeta\zeta} + u_\zeta \rho_X - \frac{\zeta u_\zeta}{2X}\rho_\zeta - \frac{\zeta \rho}{2X} u_{\zeta\zeta} + \rho_\zeta u_X - \frac{\zeta \rho_\zeta}{2X} u_\zeta \right) - \frac{C}{\delta_L^2 \rho}\left(\rho_\zeta v_\zeta + \rho_{\zeta\zeta} v \right)$$
$$- \frac{C}{a(X)}(\mathrm{i}(\omega - uk) + \frac{\varepsilon_k}{\delta_L} a'(X)(u_\zeta + \frac{\rho_\zeta}{\rho} u)) + \frac{\mathrm{i}m^2 \tilde{p}_0}{a(X)^2 \rho(\omega - uk)},$$
$$(4.15)$$

the streamwise momentum equation

$$\mathrm{i}(\omega - uk)\tilde{u}_1 = \tilde{v}_1 \frac{u_\zeta}{\delta_L} - \varepsilon_k u\left(\frac{Cu_\zeta}{\delta_L}\right)_X - \frac{\varepsilon_k C}{\delta_L}\left(-\frac{u\zeta u_{\zeta\zeta}}{2X} + u_\zeta u_X - \frac{\zeta u_\zeta^2}{2X} \right) - \frac{v}{\delta_L}\frac{Cu_{\zeta\zeta}}{\delta_L}$$
$$- \frac{C\rho_\zeta}{\rho \delta_L}(\varepsilon_k(uu_X - \frac{u\zeta}{2X} u_\zeta) + \frac{v}{\delta_L} u_\zeta) + \frac{\mathrm{i}k\tilde{p}_0}{\rho} + \frac{(\gamma - 1)\varepsilon_k C}{X\rho \delta_L}(Tu_{\zeta\zeta} + T_\zeta u_\zeta)_\zeta, \quad (4.16)$$

and the energy equation:

$$\mathrm{i}(\omega - uk)\tilde{T}_1 = \tilde{v}_1 \frac{T_\zeta}{\delta_L} - \varepsilon_k u\left(\frac{CT_\zeta}{\delta_L}\right)_X - \frac{\varepsilon_k C}{\delta_L}\left(-\frac{uT_{\zeta\zeta}\zeta}{2X} + u_\zeta T_X - \frac{\zeta u_\zeta}{2X} T_\zeta \right) - \frac{v}{\delta_L}\frac{CT_{\zeta\zeta}}{\delta_L}$$
$$- \frac{C\rho_\zeta}{\rho \delta_L}(\varepsilon_k(uT_X - \frac{u\zeta}{2X} T_\zeta) + \frac{v}{\delta_L} T_\zeta) + \frac{\mathrm{i}(\omega - uk)}{\rho}\tilde{p}_0 + \frac{\varepsilon_k C}{\delta_L \rho} u_\zeta p_X$$
$$+ \frac{\varepsilon_k(\gamma - 1)C}{\rho \delta_L X}\left[\frac{1}{Pr}(T_\zeta T)_{\zeta\zeta} + T_\zeta(u_\zeta)^2 + 2Tu_\zeta u_{\zeta\zeta} \right]. \quad (4.17)$$

We can now use the boundary layer equations for the mean flow (2.23) to simplify these equations. Transforming the mean flow boundary layer equations into the same coordinates

ζ and X we get the following equations that the mean flow satisfies to leading order

$$\varepsilon_k\left(\rho u_X - \frac{\rho\zeta u_\zeta}{2X} + u\rho_X - \frac{u\zeta\rho_\zeta}{2X}\right) + \frac{v}{\delta_L}\rho_\zeta + \frac{v_\zeta}{\delta_L}\rho + \frac{\varepsilon_k a'(X)}{a(X)}\rho u = 0, \qquad (4.18a)$$

$$\rho\left(\varepsilon_k\left(uu_X - \frac{u\zeta u_\zeta}{2X}\right) + \frac{v}{\delta_L}u_\zeta\right) = -\varepsilon_k p_X + \frac{(\gamma-1)\varepsilon_k}{X}(Tu_\zeta)_\zeta + \frac{(\gamma-1)T\varepsilon_k}{X}F_{\zeta\zeta}(X,\zeta),$$
$$(4.18b)$$

$$\rho\left(\varepsilon_k\left(uT_X - \frac{u\zeta T_\zeta}{2X}\right) + \frac{v}{\delta_L}T_\zeta\right) = \varepsilon_k u p_X + \frac{(\gamma-1)\varepsilon_k}{X}\left[\frac{(TT_\zeta)_\zeta}{Pr} + Tu_\zeta^2 + TG_{\zeta\zeta}(X,\zeta)\right].$$
$$(4.18c)$$

Our first order equations (4.15), (4.16) and (4.17) contain many terms from the ζ-derivative of these mean flow equations. We use this to simplify the equations, giving

$$i(\omega - uk)\frac{\tilde{\rho}_1}{\rho} - ik\tilde{u}_1 - \frac{\tilde{v}_{1\zeta}}{\delta_L} - \frac{\rho_\zeta \tilde{v}_1}{\rho\delta_L} = -\frac{\varepsilon_k}{\rho\delta_L}(u\rho_\zeta + \rho u_\zeta)C_X$$
$$+\frac{C}{\delta_L^2\rho}\left(\rho_\zeta v_\zeta + v_{\zeta\zeta}\rho\right) - \frac{C}{a(X)}(i(\omega - uk)) + \frac{im^2\tilde{p}_0}{a(X)^2\rho(\omega - uk)}, \qquad (4.19)$$

$$i(\omega - uk)\tilde{u}_1 = \tilde{v}_1\frac{u_\zeta}{\delta_L} - \frac{\varepsilon_k uu_\zeta}{\delta_L}C_X + \frac{Cv_\zeta u_\zeta}{\delta_L^2} + \frac{ik\tilde{p}_0}{\rho} - \frac{\varepsilon_k C(\gamma-1)}{\rho\delta_L X}(TF_{\zeta\zeta})_\zeta, \qquad (4.20)$$

$$i(\omega - uk)\tilde{T}_1 = \tilde{v}_1\frac{T_\zeta}{\delta_L} - \frac{\varepsilon_k uT_\zeta}{\delta_L}C_X + \frac{Cv_\zeta T_\zeta}{\delta_L^2} + \frac{i(\omega - uk)}{\rho}\tilde{p}_0 - \frac{\varepsilon_k C(\gamma-1)}{\rho\delta_L X}(TG_{\zeta\zeta})_\zeta. \quad (4.21)$$

Now we again use the equation of state (2.1e) to write down the density perturbation $\tilde{\rho}_1$ in terms of the temperature perturbation

$$\frac{\tilde{\rho}_1}{\rho} = \frac{\tilde{p}_0}{p} - \frac{\tilde{T}_1}{T}. \qquad (4.22)$$

We then use equations (4.20) and (4.21) to substitute for $\tilde{u}_1$ and $\tilde{T}_1$ in equation (4.19), giving an equation for $\tilde{v}_1$

$$\left(\frac{\tilde{v}_1}{\omega - uk}\right)_\zeta = \left(\frac{\varepsilon_k C_X u - C\frac{v_\zeta}{\delta_L}}{\omega - uk}\right)_\zeta + \frac{\varepsilon_k C(\gamma-1)}{\rho X(\omega - uk)}\left(\frac{(TG_{\zeta\zeta})_\zeta}{T} + \frac{k(TF_{\zeta\zeta})_\zeta}{\omega - uk}\right)$$
$$+ i\delta_L\left(\frac{C}{a} + \frac{\tilde{p}_0}{\gamma p} - \frac{\tilde{p}_0}{\gamma p}(k^2 + \frac{m^2}{a^2})\frac{(\gamma-1)T}{(\omega - uk)^2}\right). \qquad (4.23)$$

We can also use equation (4.12) to derive an equation for the correction term, $\tilde{p}_1$, to the pressure in a similar way. Expanding the equation to $\mathcal{O}(\varepsilon\delta_L)$ and $\mathcal{O}(\varepsilon\varepsilon_k)$ gives

$$\tilde{p}_{1\zeta} = \delta_L\rho\left[\mathrm{i}(\omega - uk)\tilde{v}_0 + \varepsilon_k a'(X)\mathrm{i}(\omega - uk)\tilde{u}_0 - \tilde{v}_0\frac{\varepsilon_k a'(X)}{\delta_L}u_\zeta\right],$$

but when we substitute in the leading order solutions the last two terms cancel and we find that the pressure perturbation correction term only has a $\mathcal{O}(\delta_L)$ contribution. The equation then simplifies to:

$$\tilde{p}_{1\zeta} = -\delta_L C\gamma p\frac{(\omega - uk)^2}{(\gamma - 1)T}.$$

We can then integrate this equation and write the result in the following form

$$\tilde{p}_1 = -\delta_L C\gamma p\left(\zeta\frac{(\omega - Uk)^2}{(\gamma-1)\mathcal{F}} - \int_\zeta^\infty \frac{(\omega-uk)^2}{(\gamma-1)T} - \frac{(\omega-Uk)^2}{(\gamma-1)\mathcal{F}}\,\mathrm{d}\zeta\right) + \tilde{p}_{1\infty}, \qquad (4.24)$$

where $\tilde{p}_{1\infty}$ will be matched to the acoustic pressure in region III and is expected to be order $\mathcal{O}(\varepsilon\varepsilon_k, \varepsilon\delta_L)$.

4.4.3   Region I
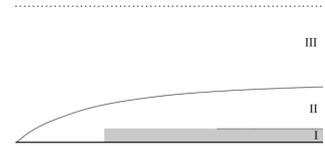
In Region I we transform into the acoustic boundary layer using the transformation $r = a(X) - \lambda y$, where $\lambda = \delta = \delta_L\sqrt{\frac{\varepsilon_k}{X}}$. Note that this variable y for region I slhould not to be confused with the different variable y used in §2.4.2 to solve for the boundary layer mean flow, as they differ by the constant factor λ. Now λ is independent of x, and we can expand the mean flow quantities about their values at the wall, e.g.

$$u = 0 + \zeta u_\zeta(0) + ... = y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0) + ...,$$

where we have used the no-slip boundary condition (2.3) to set $u(0) = 0$. From the boundary condition for the mean flow temperature (2.4) we also have that $T_\zeta(0) = 0$ so $\rho_\zeta(0) \sim p_\zeta(0) = \mathcal{O}(\delta^2)$.

By matching to the leading order equations in region II (4.14) we expect $\tilde{v}_0$, $\tilde{w}_0$ and $\tilde{p}_0$ to be $\mathcal{O}(\varepsilon)$ and $\tilde{u}_0$ and $\tilde{T}_0$ to be $\mathcal{O}(\frac{\varepsilon}{\delta_L})$. However in region II $\tilde{T}_0 \propto T_\zeta$, but when we expand T_ζ

in region I we have

$$T_\zeta = T_\zeta(0) + y\sqrt{\frac{\varepsilon_k}{X}}T_{\zeta\zeta}(0) = 0 + y\sqrt{\frac{\varepsilon_k}{X}}T_{\zeta\zeta}(0).$$

This means that the solution in region II for $\tilde{T}_0$ becomes order $\mathcal{O}(\frac{\varepsilon\sqrt{\varepsilon_k}}{\delta_L})$ when matched to region I and so we expect $\tilde{T}_0$ to be $\mathcal{O}(\frac{\varepsilon\sqrt{\varepsilon_k}}{\delta_L})$ in this region.

We can now expand the equations using these scalings for the leading order acoustic terms. The continuity equation (2.1a) now becomes

$$\mathrm{i}\left(\omega - y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)k\right)\tilde{\rho}\delta - \mathrm{i}k\rho(0)\tilde{u}\delta - \left(\rho(0) + \frac{y^2}{2}\frac{\varepsilon_k}{X}\rho_{\zeta\zeta}(0)\right)\tilde{v}_y \tag{4.25}$$

$$-\left(y\frac{\varepsilon_k}{X}\rho_{\zeta\zeta}(0)\right)\tilde{v} + \frac{\delta}{a(X)}\rho(0)\tilde{v} = \mathcal{O}(\varepsilon_k^{3/2}, \delta\sqrt{\varepsilon_k}, \delta^2)\frac{\varepsilon}{\delta_L}, \tag{4.26}$$

the streamwise momentum equation gives

$$\rho(0)\left[\mathrm{i}\left(\omega - y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)k\right)\tilde{u} - \frac{\tilde{v}}{\delta_L}\left(u_\zeta(0) + \frac{y\sqrt{\varepsilon_k}}{\sqrt{X}}u_{\zeta\zeta}(0)\right)\right] \tag{4.27}$$

$$-(\gamma - 1)\left[T(0)\tilde{u}_{yy} + \sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)\tilde{T}_y\right] = \mathcal{O}(\varepsilon_k, \delta_L)\frac{\varepsilon}{\delta_L}, \tag{4.28}$$

and the radial momentum equation gives

$$\tilde{p}_y = \mathcal{O}(\sqrt{\varepsilon_k}\delta_L, \varepsilon_k^{3/2})\varepsilon. \tag{4.29}$$

So we find that the pressure perturbation is constant to leading and first correction order and can be matched to the result from region II. Finally the energy equation (2.1d) gives

$$\rho(0)\left[\mathrm{i}\left(\omega - y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)k\right)\tilde{T} - \frac{\tilde{v}}{\delta_L}y\sqrt{\frac{\varepsilon_k}{X}}T_{\zeta\zeta}(0) + \varepsilon_k\tilde{u}T_X\right] \tag{4.30}$$

$$-(\gamma - 1)\left[T(0)\frac{\tilde{T}_{yy}}{Pr} + 2T(0)\left(\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0) + \frac{\varepsilon_k y}{X}u_{\zeta\zeta}(0)\right)\right]y = \mathcal{O}(\varepsilon_k, \delta)\frac{\varepsilon\sqrt{\varepsilon_k}}{\delta_L}. \tag{4.31}$$

Now if we just consider the leading order terms in the above equations, with $\tilde{u}_0 \sim \frac{\varepsilon}{\delta_L}$, $\tilde{v}_0 \sim \varepsilon$ and $\tilde{T}_0 \sim \frac{\varepsilon(\varepsilon_k)^{1/2}}{\delta_L}$ as discussed above so that the equations balance, we can use the leading order equation of state (4.13) to write $\tilde{\rho}_0$ in terms of $\tilde{T}_0$ and obtain the following equations

for the leading order perturbations

$$\tilde{v}_{0y} = 0,$$

$$i\omega\tilde{u}_0 - \frac{\tilde{v}_0 u_\zeta(0)}{\delta_L} = \frac{(\gamma-1)^2 T(0)^2}{\gamma p(0)}\tilde{u}_{0yy},$$

$$i\omega\tilde{T}_0 - \frac{y\tilde{v}_0 T_{\zeta\zeta}(0)\sqrt{\varepsilon_k}}{\sqrt{X}\delta_L} = \frac{(\gamma-1)^2 T(0)^2}{\gamma p(0)}\left(\frac{\tilde{T}_{0yy}}{Pr} + 2\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)\tilde{u}_{0y}\right).$$

Note that since p is approximately constant inside this region, $p \approx p(0)$, so for brevity I will just write p. Since the first equation gives $\tilde{v}_0$ constant we can then solve the other two equations analytically. Both of the other equations are second order and so need two boundary conditions; at the wall we have no-slip (2.3) $\tilde{u}_0 = 0$ and no temperature fluctuations (2.6) $\tilde{T}_0 = 0$. The other boundary condition that we apply is that the solutions cannot grow exponentially as $y \to \infty$, this is necessary for these solutions to be able to be matched to the solutions in region II. We then have at leading order in region I

$$\tilde{v}_0 = i\omega C(X), \tag{4.32a}$$

$$\tilde{u}_0 = \frac{u_\zeta(0)C(X)}{\delta_L}\left[1 - \exp\left(\frac{-y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}\right)\right], \tag{4.32b}$$

$$\tilde{T}_0 = \frac{C(X)T_{\zeta\zeta}(0)y\sqrt{\varepsilon_k}}{\delta_L\sqrt{X}} \tag{4.32c}$$

$$+ \frac{2Pr}{1-Pr}\frac{(\gamma-1)T(0)u_\zeta(0)^2 C(X)\sqrt{\varepsilon_k}}{\delta_L\sqrt{i\omega X\gamma p}}\left[\exp\left(\frac{-y\sqrt{i\omega\gamma p Pr}}{(\gamma-1)T(0)}\right) - \exp\left(\frac{-y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}\right)\right], \tag{4.32d}$$

where the square root is taken so that the real part is positive and the exponential terms decay away from the wall. Note here that the $\frac{\sqrt{\varepsilon_k}}{\delta_L}$ terms arise due to the relationship $\lambda = \delta_L\sqrt{\varepsilon_k}$, where λ is the thickness of region I.

Whereas in region II and III our leading order solutions were correct to $\mathscr{O}(\varepsilon_k)$, here our solutions (4.32) are only correct to $\mathscr{O}(\sqrt{\varepsilon_k})$. To match to region II we will also need the $\mathscr{O}(\varepsilon\sqrt{\varepsilon_k})$ correction term to $\tilde{v}_0$. We will call this term $\tilde{v}_1$ with $\tilde{v}_1 \sim \varepsilon\sqrt{\varepsilon_k}$. Expanding equation (4.25) to this order we have the following equation

$$\tilde{v}_{1y} = -\frac{ik\sqrt{\varepsilon_k}\tilde{u}_0}{\sqrt{X}},$$

this then gives

$$\tilde{v}_1 = -\frac{\mathrm{i}ku_\zeta(0)C(X)\sqrt{\varepsilon_k}}{\sqrt{X}}\left[y + \frac{(\gamma-1)T(0)}{\sqrt{\mathrm{i}\omega\gamma p}}\exp\left(\frac{-y\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}\right)\right] + \tilde{v}_{1\infty}, \qquad (4.33)$$

where $\tilde{v}_{1\infty}$ is a constant, but matching to the solution in region II gives $\tilde{v}_{1\infty} = 0$.

To find the slowly varying terms $B(X)$ and $C(X)$ we will also need the $\mathcal{O}(\varepsilon\varepsilon_k)$ solution for $\tilde{v}$ in this region. To avoid confusion with the $\tilde{v}_2$ terms in §5 I will call the $\mathcal{O}(\varepsilon\varepsilon_k)$ correction $\tilde{v}_3$. This term will satisfy the $\mathcal{O}(\varepsilon\varepsilon_k)$ expansion of equation (4.25)

$$\tilde{v}_{3y} = -y\frac{\varepsilon_k\rho_{\zeta\zeta}(0)}{X\rho(0)}\tilde{v}_0 - \mathrm{i}k\tilde{u}_1\delta_L\sqrt{\frac{\varepsilon_k}{X}} - \frac{\mathrm{i}\omega\tilde{T}_0}{T(0)}\delta_L\sqrt{\frac{\varepsilon_k}{X}}. \qquad (4.34)$$

Now before we can solve this equation for $\tilde{v}_3$ we must first solve for $\tilde{u}_1$, the $\mathcal{O}(\frac{\varepsilon\sqrt{\varepsilon_k}}{\delta_L})$ correction to $\tilde{u}_0$. Expanding equation (4.27) to this order we obtain the following equation for $\tilde{u}_1$,

$$\mathrm{i}\omega\tilde{u}_1 - \frac{\tilde{v}_1 u_\zeta(0)}{\delta_L} - \frac{(\gamma-1)^2T(0)^2}{\gamma p(0)}\tilde{u}_{1yy} = y\sqrt{\frac{\varepsilon_k}{X}}\left(\mathrm{i}ku_\zeta(0)\tilde{u}_0 + \frac{u_{\zeta\zeta}(0)\tilde{v}_0}{\delta_L}\right),$$

which has the following solution

$$\tilde{u}_1 = \frac{C(X)\sqrt{\varepsilon_k}}{\delta_L\sqrt{X}}\left(u_{\zeta\zeta}(0)y - \frac{ku_\zeta(0)^2}{4\omega}\left[\frac{\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}y^2 + 3y + \tilde{u}_{1\infty}\right]\exp\left(\frac{-y\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}\right)\right),$$

where $\tilde{u}_{1\infty}$ is a constant. Applying the no-slip condition at $y = 0$ gives $\tilde{u}_{1\infty} = 0$.

We can now solve equation (4.34) for $\tilde{v}_3$,

$$\begin{aligned}\tilde{v}_{3y} = &-\frac{\varepsilon_k\mathrm{i}kCyu_{\zeta\zeta}(0)}{X} + \frac{\mathrm{i}k^2u_\zeta(0)^2C\varepsilon_k}{4\omega X}\left[\frac{\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}y^2 + 3y\right]\exp\left(\frac{-y\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}\right)\\ &-\frac{\varepsilon_k}{X}\frac{2Pr}{1-Pr}\frac{(\gamma-1)Cu_\zeta(0)^2\sqrt{\mathrm{i}\omega}}{\sqrt{\gamma p}}\left[\exp\left(\frac{-y\sqrt{\mathrm{i}\omega\gamma pPr}}{(\gamma-1)T(0)}\right) - \exp\left(\frac{-y\sqrt{\mathrm{i}\omega\gamma p}}{(\gamma-1)T(0)}\right)\right].\end{aligned} \qquad (4.35)$$

4.5   Matching and dispersion relation

Now that we have our solutions in each region we can match them and derive the dispersion relation. First we consider the leading order asymptotics to obtain the dispersion relation for k. Given that the leading order pressure term $\tilde{p}_0$ is constant in regions I and II we have

$$\tilde{p}_0|_{r=a} = B(X)J_m(\alpha a) = \tilde{p}_0|_{y=0},$$

where $\alpha^2 = (\omega - Uk)^2 - k^2$ as before. We can now apply the impedance boundary condition at $y = 0$ in region I

$$\tilde{p}_0|_{y=0} = Z\tilde{v}_0|_{y=0} = Zi\omega C(X).$$

We have already matched $\tilde{v}_0$ between regions I and II so we only need to match between regions II and III to get an expression for $C(X)$ in terms of $B(X)$. At leading order this matching simply involves setting $\tilde{v}_0|_{\zeta \to \infty} = \tilde{v}_0|_{r=a}$ which gives

$$iC(\omega - Uk) = \tilde{v}_0|_{r=a} = \frac{-\alpha B(X)J_m'(\alpha a)}{\rho i(\omega - Uk)}. \tag{4.36}$$

Putting this all together we have

$$\tilde{p}_0|_{r=a} = \frac{Z\omega}{(\omega - Uk)}\tilde{v}_0|_{r=a},$$

so $Z_{\text{eff}} = \frac{Z\omega}{(\omega - Uk)}$ and the dispersion relation is

$$J_m(\alpha a) = \frac{Zi\omega\alpha J_m'(\alpha a)}{\rho(\omega - Uk)^2}, \tag{4.37}$$

so that k depends on X only through the outer mean flow terms $U(X)$ and $\rho(X)$. This is the same dispersion relation as for the Myers boundary condition, which makes sense as the leading order solution corresponds to an infinitely thin boundary ($\delta_L \to 0$) without nonparallel effects ($\varepsilon_k \to 0$). However if we now try to match to order $\mathscr{O}(\varepsilon\sqrt{\varepsilon_k})$ we find that the contribution from the $\tilde{v}_1$ term in region I must be included with the leading order terms, otherwise it cannot be matched. This is because there are no $\mathscr{O}(\varepsilon\sqrt{\varepsilon_k})$ correction terms in regions II and III, and including terms of this order in these regions is equivalent to adjusting the slowly varying terms $B(X)$ and $C(X)$, which drop out during the matching. Including the $\tilde{v}_1$ term in the matching gives the following modified dispersion relation due to viscous effects in the acoustic boundary layer

$$\frac{Z\omega}{(\omega - Uk)}\tilde{v}_0|_{r=a} = \tilde{p}_0|_{r=a}\left[1 - \frac{ku_\zeta(0)(\gamma - 1)T(0)}{\omega}\sqrt{\frac{\varepsilon_k}{i\omega\gamma p(0)X}}\right]^{-1} + \mathscr{O}(\varepsilon_k, \delta_L).$$

We can improve the accuracy of this dispersion relation by considering the correction terms when matching to region III. We find that the expansion of $\tilde{p}_0$ in region III involves

only $\mathscr{O}(\varepsilon\delta_L)$ and not $\mathscr{O}(\varepsilon\varepsilon_k)$ terms

$$\tilde{p}_{0III}(a(X) - \delta_L\zeta) \approx \tilde{p}_{0III}(a(X)) - \delta_L\zeta\tilde{p}_{0IIIr}(a(X)).$$

Since δ_L and ε_k are independent small parameters, this means that the $\mathscr{O}(\varepsilon\delta_L)$ correction terms in region II must match to the expansion of the leading order terms, and will give a correction to the dispersion relation. While the $\mathscr{O}(\varepsilon\varepsilon_k)$ correction terms in region II will match to the first order correction term, $\tilde{p}_1$, in region III, and so will contribute to the slowly varying term $B(X)$. We then have

$$\tilde{p}_{0III}(a(X)) - \delta_L\zeta\tilde{p}_{0IIIr}(a(X)) = \tilde{p}_{0II} + \tilde{p}_{1II}^{\delta_L}|_{\zeta\to\infty},$$

where $\tilde{p}_{1II}^{\delta_L}|_{\zeta\to\infty}$ is the $\mathscr{O}(\varepsilon\delta_L)$ terms of $\tilde{p}_1$ (4.24) in region II for large ζ

$$\tilde{p}_{1II}^{\delta_L}|_{\zeta\to\infty} = -\delta_L C\gamma p\zeta\frac{(\omega - Uk)^2}{(\gamma-1)\mathcal{T}} + \tilde{p}_{1\infty}^{\delta_L},$$

where $\tilde{p}_{1\infty}^{\delta_L}$ is the $\mathscr{O}(\varepsilon\delta_L)$ constant term in the solution (4.24). Matching this to the solution in region III and using the leading order result (4.36) for $C(X)$ gives

$$-\delta_L\zeta BJ_m'(\alpha a)\alpha = -\delta_L C\gamma p\zeta\frac{(\omega - Uk)^2}{(\gamma-1)\mathcal{T}} + \tilde{p}_{1\infty}^{\delta_L} = -\delta_L\zeta BJ_m'(\alpha a)\alpha + \tilde{p}_{1\infty}^{\delta_L},$$

and so we must have $\tilde{p}_{1\infty}^{\delta_L} = 0$. We can then use the solution to $\tilde{p}_1$ in region II (4.24) for small ζ to match to region I. This gives

$$\tilde{p}_0 + \tilde{p}_{1II}^{\delta_L}|_{\zeta=0} = \tilde{p}_0 + \tilde{p}_{1I},$$

where $\tilde{p}_{1I}$ is the correction in region I to the leading order pressure perturbation $\tilde{p}_0$, but both $\tilde{p}_0$ and $\tilde{p}_{1I}$ are constant so we can now write the pressure perturbation at the wall as

$$\tilde{p}|_{y=0} = B(X)J_m(\alpha a) + \delta_L C(X)\gamma p\left(\int_0^\infty \frac{(\omega - uk)^2}{(\gamma-1)T} - \frac{(\omega - Uk)^2}{(\gamma-1)\mathcal{T}}\,d\zeta\right).$$

To apply the impedance boundary condition at the wall we first have to apply a similar procedure to match $\tilde{v}$ between each region. From (4.23), in region II we have

$$\left(\frac{\tilde{v}_{1II}^{\delta_L}}{\omega - uk}\right)_\zeta = i\delta_L\left(\frac{C}{a} + \frac{\tilde{p}_0}{\gamma p} - \frac{\tilde{p}_0}{\gamma p}(k^2 + \frac{m^2}{a^2})\frac{(\gamma-1)T}{(\omega - uk)^2}\right),$$

which gives

$$
\tilde{v}_{1II}^{\delta_L} = i\delta_L(\omega - uk)\zeta \left(\frac{C}{a} + \frac{\tilde{p}_0}{\gamma p} - \frac{\tilde{p}_0}{\gamma p} \left(k^2 + \frac{m^2}{a^2} \right) \frac{(\gamma-1)\mathcal{F}}{(\omega-Uk)^2} \right)
$$
$$
+ i\delta_L(\omega - uk)\frac{\tilde{p}_0}{\gamma p} \left(k^2 + \frac{m^2}{a^2} \right) \left(\int_\zeta^\infty \frac{(\gamma-1)T}{(\omega-uk)^2} - \frac{(\gamma-1)\mathcal{F}}{(\omega-Uk)^2} \, d\zeta \right) + \tilde{v}_{1\infty}^{\delta_L}.
$$

Matching to region III and using the leading order results we find again that $\tilde{v}_{1\infty}^{\delta_L} = 0$. We can now match to $\tilde{v}$ in region I which has only the $\mathcal{O}(\varepsilon\sqrt{\varepsilon_k})$ viscous correction term mentioned above and no $\mathcal{O}(\varepsilon\delta_L)$ terms. For small ζ the solution in region II is

$$
\tilde{v}_{II} \sim iC \left(\omega - ku_\zeta(0)y\sqrt{\frac{\varepsilon_k}{X}} - ku_{\zeta\zeta}(0)\frac{y^2\varepsilon_k}{2X} \right)
$$
$$
+ i\delta_L\omega\frac{\tilde{p}_0}{\gamma p} \left(k^2 + \frac{m^2}{a^2} \right) \int_0^\infty \frac{(\gamma-1)T}{(\omega-uk)^2} - \frac{(\gamma-1)\mathcal{F}}{(\omega-Uk)^2} \, d\zeta.
$$

The first constant term will match to $\tilde{v}_0$ in region I, the second linear term will match to $\tilde{v}_1$ in region I and the quadratic term will match to the quadratic term in $\tilde{v}_3$ in region I. The integral term is now constant and only provides a $\mathcal{O}(\varepsilon\delta_L)$ correction to the constant term in $\tilde{v}_0$.

We can now apply the impedance boundary condition at the wall and use the result (4.36) to cancel the slowly varying terms $B(X)$ and $C(X)$. This then gives a correction to the dispersion relation due to the inviscid shear in region II in terms of two integrals

$$
\delta I_1(X) = \delta_L \int_0^\infty \frac{(\gamma-1)T}{(\omega-uk)^2} - \frac{(\gamma-1)\mathcal{F}}{(\omega-Uk)^2} \, d\zeta, \tag{4.38a}
$$

$$
\delta I_2(X) = \delta_L \int_0^\infty \frac{(\omega-uk)^2}{(\gamma-1)T} - \frac{(\omega-Uk)^2}{(\gamma-1)\mathcal{F}} \, d\zeta, \tag{4.38b}
$$

these correspond to the $\mathcal{O}(\delta_L)$ change in $\tilde{v}_1$ and $\tilde{p}_1$ respectively across region II. When including these terms in the matching we now obtain the full dispersion relation to $\mathcal{O}(\delta_L)$

$$
J_m(\alpha a) \left[1 - \frac{Zi\omega}{\gamma p(0)} \left(k^2 + \frac{m^2}{a^2} \right) \delta I_1 \right] = \frac{i\alpha J_m'(\alpha a) \left[Z\left(\omega - \frac{ku_\zeta(0)(\gamma-1)T(0)\sqrt{\varepsilon_k}}{\sqrt{i\omega\gamma p(0)X}} \right) + i\gamma p(0)\delta I_2 \right]}{\rho(\omega-Uk)^2}
$$

$$
\tag{4.39}
$$

This dispersion relation can also be written in terms of the effective impedance Z_{eff}

$$Z_{\text{eff}} = \frac{\tilde{p}_0(a)}{\tilde{v}_0(a)} = \frac{\frac{Z\omega}{\omega - Uk}\left[1 - \frac{ku_\zeta(0)(\gamma-1)T(0)\sqrt{\varepsilon_k}}{\omega\sqrt{i\omega\gamma p(0)X}} + \frac{i\gamma p(0)\delta I_2}{Z\omega}\right]}{\left[1 - \frac{Zi\omega}{\gamma p(0)}\left(k^2 + \frac{m^2}{a^2}\right)\delta I_1\right]},\tag{4.40}$$

where $p(0)$, $T(0)$ and $u_\zeta(0)$ are the mean flow boundary layer terms evaluated at the wall and all other mean flow terms are evaluated in the centre of the duct. Note there are no $\mathcal{O}(\varepsilon_k)$ terms here as these will be taken into account by the slowly varying function $B(X)$. Which will be found in the next section by deriving a compatibility condition. Also the $\varepsilon a' \tilde{u}$ correction from the transformation of v in region II does not affect the dispersion relation because $\tilde{u}_0 \to 0$ as $\zeta \to \infty$.

We can compare this effective impedance to previous results. If we consider the modified Myers boundary condition [5], which takes into account the boundary layer shear but assumes viscosity is negligible and the duct is parallel, this results in the local effective impedance given in equation (1.6)

$$Z_{\text{eff}}^{\text{Brambley}} = \frac{Z\omega}{\omega - Uk}\frac{1 - \frac{i(\omega - Uk)^2}{Z\omega}\delta I_0^B}{1 + \frac{i\omega Z(k^2 + m^2)}{(\omega - Uk)^2}\delta I_1^B}.$$

If we assume $a = 1$, the mean flow temperature and pressure don't vary within the boundary layer and we ignore the viscous $\mathcal{O}\left(\sqrt{\frac{\varepsilon_k}{X}}\right)$ term we can see that

$$\delta I_0^B = -\frac{\delta I_2}{(\omega - Uk)^2},$$
$$\delta I_1^B = -\delta I_1(\omega - Uk)^2,$$

and the two effective impedances agree.

We can also compare the slowly varying effective impedance (4.40) to previous work on a three layer model [17] that included the effect of viscosity. This previous work used the assumption $\xi\delta_L^3 \sim 1/Re$, with $\xi \sim \mathcal{O}(1)$ to derive a local parallel three layer model. Although this previous work assumed the flow was parallel, if we set $\xi = \frac{U}{x\delta_L}$ we have the same scaling regime as in this thesis. We can then write down the effective impedance in terms of the notation used in this thesis for $a = 1$

$$Z_{\text{eff}}^{\text{Khamis}} = \frac{Z\omega}{\omega - Uk}\left(\frac{1 - \frac{(\gamma-1)T(0)ku_\zeta(0)\sqrt{\varepsilon_k}}{\omega\sqrt{i\omega X}} + \frac{i}{Z\omega}\delta I_2}{1 - i(k^2 + m^2)\omega Z\delta I_1}\right) + \mathcal{O}(\varepsilon_k, \sqrt{\varepsilon_k}\delta_L).\tag{4.41}$$

We can see that at this order the dispersion relation above agrees with what we derived for the slowly varying duct.[2] However at higher orders the corrections to this dispersion relation will not agree, as the assumption of parallel flow means that this previous work only gives a local solution and does not take into account the variation of the boundary layer thickness or the mean flow normal velocity. Here these effects are taken into account by the slowly varying term $B(X)$ and by integrating $k(X)$ over the duct (4.2), rather than evaluating $k(X)x$ locally, to give a global solution.

4.6 Slowly varying part - compatibility condition

To solve for the slow non-parallel variations, we must obtain a solvability condition. To do this we take the inner product of the unknown first order correction solution with the known leading order solution in each region, matching between the regions, and then use the boundary conditions to get an equation only in terms of the slow variation. We will want to match our $\mathcal{O}(\varepsilon_k)$ solution from region III to the wall to apply the boundary condition. To do this we must use the parts of the solutions in regions II and I that were not matched by the leading order solution.

In region III we have the $\mathcal{O}(\varepsilon_k)$ correction $\tilde{p}_1$ to the leading order pressure perturbation given by equation (4.7)

$$\tilde{p}_{1rr} + \frac{\tilde{p}_{1r}}{r} + \tilde{p}_1 \left(\alpha^2 - \frac{m^2}{r^2} \right) = R_{III} = 2ik\varepsilon_k \tilde{p}_{0X} + \frac{2i(\omega - Uk)}{(\gamma - 1)\mathcal{F}} \left(V\tilde{p}_{0r} + U\varepsilon_k \tilde{p}_{0X} \right)$$

$$+ \varepsilon_k i\tilde{p}_0 \left[k_X + \frac{(\omega - Uk)}{(\gamma - 1)\mathcal{F}} \left(U_X + \frac{4V_r}{\varepsilon_k} \right) - \frac{k_X U^2}{(\gamma - 1)\mathcal{F}} + 2k^2 \frac{U_X - \frac{V_r}{\varepsilon_k}}{\omega - Uk} + \frac{(1 - 2\gamma)U\mathcal{F}_X(\omega - Uk)}{(\gamma - 1)^2 \mathcal{F}^2} \right]$$

Given that the left hand side of this equation is a Bessel's equation we multiply this by $J_m(\alpha r)r$ and integrate from 0 to $a(X) - \delta_L^\beta \eta$, where $0 < \beta < 1$ and η is a matching variable that lets us match to region II. Note that as we are only matching up to order $\mathcal{O}(\varepsilon \varepsilon_k)$ we will ignore any $\mathcal{O}(\varepsilon \varepsilon_k \delta_L)$ correction terms and so we may set $\eta = 0$. The integral of the left hand side may be integrated by parts so that the only remaining terms are the boundary terms, so we have

$$\left[\tilde{p}_{1r} J_m(\alpha r)r - \tilde{p}_1 \alpha J'_m(\alpha r)r \right]_0^{a(X) - \delta_L^\beta \eta} = \varepsilon_k \int_0^{a(X) - \delta_L^\beta \eta} R_{III} J_m(\alpha r)r \, dr. \tag{4.42}$$

---

[2]Note that $p_\zeta \approx 0$ and $P = \frac{1}{\gamma}$ for a parallel duct so $\gamma p(0) \approx 1$.

But we can use equation (4.6c) to substitute $\tilde{p}_{1r}$ for $\tilde{v}_1$ and known leading order terms

$$\tilde{p}_{1r} = -\rho i(\omega - Uk)\tilde{v}_1 + R_v = -\rho i(\omega - Uk)\tilde{v}_1 - \varepsilon_k \rho U \tilde{v}_{0X} - \frac{V\tilde{p}_0}{i(\omega - Uk)}(\alpha^2 - \frac{m^2}{r^2}),$$

using this in equation (4.42) we obtain:

$$-\rho i(\omega - Uk)\tilde{v}_1|_{r=a} a J_m(\alpha a) - \tilde{p}_1|_{r=a}\alpha a J_m'(\alpha a)$$
$$= a J_m(\alpha a)\left(\varepsilon_k \rho U \tilde{v}_{0X}|_{r=a} + \frac{V\tilde{p}_0|_{r=a}}{i(\omega - Uk)}(\alpha^2 - \frac{m^2}{a^2})\right) + \varepsilon_k \int_0^a R_{III} J_m(\alpha r) r \, dr. \tag{4.43}$$

Now in regions I and II the remaining part of $\tilde{p}_1$ is constant since the only non-constant contribution to $\tilde{p}_1$ was the $\mathcal{O}(\varepsilon\delta_L)$ contribution (4.24) in region II which was used to match the leading order solutions and derive the dispersion relation. However $\tilde{v}_1$ has non-constant parts that were not used for the matching, so we need to match these remaining parts across the regions to apply the impedance boundary condition at the wall.

To match $\tilde{v}_1$ through region II we first note that $\tilde{v}_1|_{r=a} = \tilde{v}_1|_{\zeta\to\infty} + \varepsilon_k a'(X)\tilde{u}_0|_{\zeta\to\infty}$ from the transformation of v, where $\tilde{v}_1|_{r=a}$ is the solution in region III at $r = a$ and $\tilde{v}_1|_{\zeta\to\infty}$ and $\tilde{u}_0|_{\zeta\to\infty}$ are the solutions in region II for large ζ. But $\tilde{u}_0|_{\zeta\to\infty} = 0$ so $\tilde{v}_1|_{r=a} = \tilde{v}_1|_{\zeta\to\infty}$. So we can integrate the equation for $\tilde{v}_1$ in region II (4.23), ignoring the $\mathcal{O}(\varepsilon\delta_L)$ terms as they have been dealt with in the dispersion relation, to match through the region. This gives

$$\left[\frac{\tilde{v}_1}{\omega - uk}\right]_0^\infty = \left[\frac{\varepsilon_k C_X u - \frac{Cv_\zeta}{\delta_L}}{\omega - uk}\right]_0^\infty + \frac{\varepsilon_k C(\gamma - 1)}{X}\int_0^\infty \left(\frac{(TG_{\zeta\zeta})_\zeta}{\rho T(\omega - uk)} + \frac{k(TF_{\zeta\zeta})_\zeta}{\rho(\omega - uk)^2}\right) d\zeta. \tag{4.44}$$

At the wall we have $u(0) = 0$ from the no-slip condition. However using the boundary layer mean flow continuity equation (4.18a) at the wall, and using $\zeta = u = u_X = v = 0$ at the wall from the no-slip condition, we must also have $v_\zeta(0) = 0$. Similarly, using equation (4.18a) for $\zeta \to \infty$ we have

$$v_\zeta(\infty) = -\frac{\delta_L \varepsilon_k}{\rho}\left(\frac{a'}{a}\rho U + (\rho U)_X\right) = -\frac{\delta_L \varepsilon_k}{a\rho}\left(\frac{\dot{m}}{\pi a}\right)_X = \frac{\varepsilon_k \delta_L a'(X) U}{a(X)},$$

where we have used the continuity equation (2.9) for the outer mean flow to simplify the expression. These results mean that equation (4.44) simplifies to

$$\tilde{v}_1|_{\zeta\to\infty} = \frac{(\omega - Uk)\tilde{v}_1|_{\zeta\to 0}}{\omega} + \varepsilon_k C_X U - \frac{\varepsilon_k C U a_X}{a} + \varepsilon_k C(X)S_{bl}(X), \tag{4.45}$$

where

$$S_{bl}(X) = \frac{(\gamma-1)(\omega-Uk)}{X}\int_0^\infty \left(\frac{(TG_{\zeta\zeta})_\zeta}{\rho T(\omega-uk)} + \frac{k(TF_{\zeta\zeta})_\zeta}{\rho(\omega-uk)^2}\right)d\zeta, \qquad (4.46)$$

is a term that arises due to the functions F and G which allow for different boundary layer profiles to be used.

We can now match $\tilde{v}_1|_{\zeta=0}$ to $\tilde{v}_3$, the $\mathscr{O}(\varepsilon\varepsilon_k)$ solution in region I. Integrating the equation for $\tilde{v}_3$ (4.35) from 0 to ∞ in region I and ignoring the quadratic term that has already been matched when solving for the dispersion relation, we have

$$\tilde{v}_3|_{y\to\infty} = \tilde{v}_3|_{y=0} + \varepsilon_k C(X)S_{viscous}(X), \qquad (4.47)$$

where

$$S_{viscous}(X) = \frac{u_\zeta(0)^2(\gamma-1)^2 T(0)^2}{\gamma pX}\left(\frac{5k^2}{4\omega^2} - \frac{2\sigma}{T(0)(1+\sigma)}\right),$$

and $\sigma = \sqrt{Pr}$. This term arises due to the effect of viscosity in the inner acoustic boundary layer region.

Now if we substitute these results into equation (4.43) we have

$$-\rho i\frac{(\omega-Uk)^2}{\omega}\tilde{v}_1|_{y=0}aJ_m(\alpha a) - \tilde{p}_1|_{y=0}\alpha a J'_m(\alpha a)$$

$$= \rho\varepsilon_k i(\omega-Uk)aJ_m(\alpha a)\left[C\left(\frac{\omega-Uk}{\omega}S_{viscous} + S_{bl} - \frac{Ua_X}{a}\right) + C_X U\right] \qquad (4.48)$$

$$+ aJ_m(\alpha a)\left(\varepsilon_k\rho U\tilde{v}_{0X}|_{r=a} + \frac{V\tilde{p}_0|_{r=a}}{i(\omega-Uk)}\left(\alpha^2 - \frac{m^2}{a^2}\right)\right) + \varepsilon_k\int_0^a R_{III}J_m(\alpha r)r\,dr,$$

but since we now have both $\tilde{v}_1$ and $\tilde{p}_1$ evaluated at the wall we can use the impedance boundary condition $\tilde{p}_1(0) = Z\tilde{v}_1(0)$. This means that the left hand side of this equation becomes

$$-a\tilde{v}_1|_{y=0}\left[Z\alpha J'_m(\alpha a) + \frac{\rho i(\omega-Uk)^2}{\omega}J_m(\alpha a)\right].$$

Since we are only interested in the result up to order $\mathscr{O}(\varepsilon\varepsilon_k)$, and $\tilde{v}_1 \sim \mathscr{O}(\varepsilon\varepsilon_k)$, we can apply the leading order dispersion relation (4.37), to find that this expression cancels. So

equation (4.48) becomes

$$
\begin{aligned}
0 = \rho \mathrm{i}(\omega - Uk) a J_m(\alpha a) & \left[C \left(\frac{\omega - Uk}{\omega} S_{viscous} + S_{bl} - \frac{U a_X}{a} \right) + C_X U \right] \\
& + a J_m(\alpha a) \left(\rho U \tilde{v}_{0X}|_{r=a} + \frac{V \tilde{p}_0|_{r=a}}{\varepsilon_k \mathrm{i}(\omega - Uk)} \left(\alpha^2 - \frac{m^2}{a^2} \right) \right) + \int_0^a R_{III} J_m(\alpha r) r \, \mathrm{d}r,
\end{aligned}
\tag{4.49}
$$

which is a first order differential equation for the slowly varying amplitude $C(X)$.

   To simplify this equation we can first use the expression for the outer mean flow normal velocity (2.19)

$$
V = \frac{\varepsilon_k a_X U r}{a}.
$$

From the leading order matching of the solutions in region II and III (4.36) we also have

$$
C = \frac{B \alpha J_m'(\alpha a)}{\rho (\omega - Uk)^2}.
\tag{4.50}
$$

Finally note that R_{III} contains terms involving the leading order pressure $\tilde{p}_0$ and its first derivatives $\tilde{p}_{0r}$ and $\tilde{p}_{0X}$. This means that the term involving the integral of R_{III} contains two different integrals of Bessel functions for which the following identity may be used [9, Eq. 10.22.5]

$$
\int_0^a r J_m(\alpha r)^2 \, \mathrm{d}r = \frac{a^2}{2} \left(J_m^2(\alpha a) - J_{m-1}(\alpha a) J_{m+1}(\alpha a) \right).
\tag{4.51}
$$

The second Bessel function integral can then be evaluated by integrating by parts and using the above identity to give

$$
\int_0^a r^2 J_m(\alpha r) J_m'(\alpha r) \, \mathrm{d}r = \frac{a^2}{2\alpha} \left(J_{m-1}(\alpha a) J_{m+1}(\alpha a) \right),
\tag{4.52}
$$

but we can use the identity for the derivative of a Bessel's function

$$
J_{m-1}(x) J_{m+1}(x) = \frac{m^2}{x^2} J_m(x)^2 - J_m'(x)^2,
$$

and we can then use the dispersion relation to write

$$
J_{m-1}(a\alpha) J_{m+1}(a\alpha) = J_m(a\alpha)^2 \left(\frac{m^2}{a^2 \alpha^2} + \frac{\rho^2 (\omega - Uk)^2}{\alpha^2 Z_{eff}^2} \right).
$$

This gives

$$\int_0^a R_{III} J_m(\alpha r) r\, dr = B_X(X) J_m(a\alpha)^2 S_{meanflow}(X) + B(X) J_m(a\alpha)^2 S_{geometry}(X),$$

where

$$S_{meanflow}(X) = \mathrm{i}a^2 \left(k + \frac{(\omega - Uk)U}{(\gamma-1)\mathcal{T}} \right) \left(1 - \frac{m^2}{a^2\alpha^2} - \frac{\rho^2(\omega-Uk)^2}{\alpha^2 Z_{eff}^2} \right),$$

is a term that is due to the outer mean flow in the duct, and

$$
\begin{aligned}
S_{geometry}(X) = {}& \mathrm{i}a^2 \left(\frac{\alpha_X k}{\alpha} + \frac{(\omega - Uk)}{(\gamma-1)\mathcal{T}} \left(\frac{Ua_X}{a} + \frac{U\alpha_X}{\alpha} \right) \right) \left(\frac{m^2}{a^2\alpha^2} + \frac{\rho^2(\omega-Uk)^2}{\alpha^2 Z_{eff}^2} \right) \\
&+ \frac{a^2}{2} \left(1 - \frac{m^2}{a^2\alpha^2} - \frac{\rho^2(\omega-Uk)^2}{\alpha^2 Z_{eff}^2} \right) \left[\mathrm{i}k_X - \frac{\mathrm{i}k_X U^2}{(\gamma-1)\mathcal{T}} \right. \\
&\left. + \frac{\mathrm{i}(\omega-Uk)}{(\gamma-1)\mathcal{T}} (U_X + 4\frac{Ua_X}{a}) + 2\mathrm{i}k^2 \frac{U_X - \frac{Ua_X}{a}}{\omega - Uk} + \frac{(1-2\gamma)U\mathcal{T}_X \mathrm{i}(\omega - Uk)}{(\gamma-1)^2\mathcal{T}^2} \right],
\end{aligned}
$$

is a term due to the variation in the radius of the duct, $a(X)$.

Putting this all together in equation (4.49) we have

$$
\begin{aligned}
B_X & \left(S_{meanflow} J_m(a\alpha)^2 + \frac{2\mathrm{i}\alpha a U J_m(a\alpha)J_m'(a\alpha)}{\omega - Uk} \right) = \\
& -B \left(S_{geometry} J_m(a\alpha)^2 + \mathrm{i}\alpha a J_m(a\alpha)J_m'(a\alpha) \left(\frac{S_{viscous}}{\omega} + \frac{S_{bl} - \frac{Ua_X}{a}}{\omega - Uk} \right) \right) \\
& + \mathrm{i}\alpha a B \left(\frac{U J_m(a\alpha)J_m'(a\alpha)}{\omega - Uk} \left(\frac{a_X}{a} + \frac{2\rho_X}{\rho} - \frac{3(Uk)_X}{\omega - Uk} \right) + 2U(\alpha a)_X \frac{J_m(a\alpha)^2(1 - \frac{m^2}{a^2\alpha^2})}{(\omega - Uk)} \right).
\end{aligned}
$$

Dividing out $J_m(a\alpha)^2$ and using the dispersion relation we get

$$
\begin{aligned}
B_X & \left(S_{meanflow} + \frac{2\rho a U}{Z_{eff}} \right) = -B \left(S_{geometry} + \frac{a\rho}{Z_{eff}} \left(\frac{(\omega-Uk)S_{viscous}}{\omega} + S_{bl} - \frac{Ua_X}{a} \right) \right) \\
& + B \left(\frac{\rho a U}{Z_{eff}} \left(\frac{a_X}{a} + \frac{2\rho_X}{\rho} - \frac{3(Uk)_X}{\omega - Uk} \right) + 2 \left(\frac{U\alpha_X}{\alpha} + \frac{Ua_X}{a} \right) \frac{\mathrm{i}(a^2\alpha^2 - m^2)}{(\omega - Uk)} \right),
\end{aligned}
$$

which we can solve to obtain $B(X)$

$$B(X) = B_0 \exp\left[-\int_{X_0}^X \frac{\tilde{S}_{geometry} Z_{eff} + a\rho\left(\frac{(\omega - Uk)S_{viscous}}{\omega} + S_{bl}\right)}{S_{meanflow} Z_{eff} + 2\rho a U}\, \mathrm{d}X \right],\qquad(4.53)$$

where

$$\tilde{S}_{geometry} = S_{geometry} - \left(\frac{\rho a U}{Z_{eff}}\left(\frac{2a_X}{a} + \frac{2\rho_X}{\rho} - \frac{3(Uk)_X}{\omega - Uk}\right) + 2\left(\frac{U\alpha_X}{\alpha} + \frac{Ua_X}{a}\right)\frac{\mathrm{i}(a^2\alpha^2 - m^2)}{(\omega - Uk)}\right)$$

$$(4.54)$$

is the total contribution due to the slowly varying geometry of the duct. If the duct is cylindrical, i.e. $a'(x) = 0$ then this term will vanish.

To find the constants B_0 for each mode we can consider the Green's function solution for a point source at $X = X_0$ in a small region near the source and use the Briggs-Bers method to distinguish between upstream and downstream modes. We can then match the resulting Green's function back to our WKB solution which will give the mode amplitudes B_0. This procedure is covered in the following two sections.

4.7   Green's function

So far we have only considered the case of a single mode but to solve for the constants B_0 we need to consider a particular initial condition or forcing. Here we will consider the case of a point source forcing at $x = x_0$ starting at time $t = 0$ with no acoustics prior to that. The solution to this will give us the Green's function for the problem.

We begin by considering a harmonic point mass source of the form

$$q(t) = \mathrm{Re}\left[q\delta(x - x_0)\delta(r - r_0)H(t)e^{\mathrm{i}\omega_f t}\frac{\delta(\theta)}{r} \right],\qquad(4.55)$$

where $H(t)$ is a Heaviside function, $\delta(\cdot)$ is a delta function and ω_f is the forcing frequency. Note the difference between the time varying forcing, always written $q(t)$, and the constant forcing strength q. The point source is located at $(r_0, x_0, 0)$, the source angle may be set to $\theta = 0$ without loss of generality due to the symmetry of the problem. It should be noted that once the solution for this point forcing has been obtained it is possible to construct the solution for any other forcing function by using the linearity of the equations and integrating the Green's function solution over the forcing.

Given this form for the source, the continuity equation will now be

$$\rho_t + (\rho u)_x + \frac{1}{r}(r\rho v)_r + (\rho w)_\theta = \dot{q}(t). \tag{4.56}$$

We first assume that the source is sufficiently far from the wall such that the source is in region III. This means that we only need to solve in the outer region. We now need to transform equation (4.56) to be in terms of ω, k and m. Since we are only interested in the solution near the source, as this is where the constants B_0 are set, we transform to a small local region near x_0 by setting $x = x_0 + \eta$ where $|\eta| \ll 1$. This means that all slowly varying streamwise terms may be approximated by their values at the source location $X_0 = \varepsilon_k x_0$. Given this, we then find that inside this region the WKB solution simplifies to a Fourier transform in x

$$\tilde{u} = \tilde{u}_0(X, r)e^{-i\left[\int_{x_0}^x k(X)\,dx - \omega t + m\theta\right]} \approx \tilde{u}_0(X_0, r)e^{-i[k\eta - \omega t + m\theta]}.$$

We can then use this, along with the Fourier transform of a δ-function, to transform the expression for the source, using a Fourier series in θ, a Fourier transform in x and for causality a Laplace transform in t. This then gives the transformed continuity equation (4.56) as

$$\tilde{p}_{0rr} + \frac{\tilde{p}_{0r}}{r} + \tilde{p}_0\left(\alpha^2 - \frac{m^2}{r^2}\right) = \frac{\omega(\omega - Uk)q\delta(r - r_0)}{2\pi r_0}\frac{-i}{\omega - \omega_f},$$

which is a forced Bessel's equation, with a δ-function forcing. This has a solution in terms of Bessel functions of order m in the region $r < r_0$ and $r > r_0$. We use the condition of regularity at $r = 0$, continuity and a jump condition at $r = r_0$, and the effective impedance at $r = a(X)$ to match the solutions and apply boundary conditions. This gives the following solution

$$\tilde{p}_0 = \begin{cases} \tilde{q}\left(\frac{\rho(\omega - Uk)Y_m(\alpha a) - i\alpha Y_m'(\alpha a)Z_{eff}}{\rho(\omega - Uk)J_m(\alpha a) - i\alpha J_m'(\alpha a)Z_{eff}}J_m(\alpha r_0) - Y_m(\alpha r_0)\right)J_m(\alpha r), & r < r_0 \\ \tilde{q}\left(\frac{\rho(\omega - Uk)Y_m(\alpha a) - i\alpha Y_m'(\alpha a)Z_{eff}}{\rho(\omega - Uk)J_m(\alpha a) - i\alpha J_m'(\alpha a)Z_{eff}}J_m(\alpha r_0)J_m(\alpha r) - J_m(\alpha r_0)Y_m(\alpha r)\right), & r > r_0 \end{cases}$$

where $\tilde{q} = \frac{-iq}{\omega - \omega_f}\frac{\omega(\omega - Uk)}{4}$ and $Z_{eff} = \frac{\tilde{p}|_{r=a}}{\tilde{v}|_{r=a}}$ is given by the dispersion relation (4.40).

Now this expression can only have singularities at the roots of the dispersion relation $\rho(\omega - Uk)J_m(\alpha a) = i\alpha J_m'(\alpha a)Z_{eff}$ and at $\alpha = 0$. However if we expand near $\alpha = 0$ we find that these singularities are removable. To show this we first note the series expansions of

$J_m(\alpha)$ and $Y_m(\alpha)$

$$J_m(\alpha) = \alpha^m \sum_{n=0}^{\infty} \alpha^{2n} b_n,$$

$$Y_m(\alpha) = \frac{2}{\pi} \ln\left(\frac{\alpha}{2}\right) J_m(\alpha) + \alpha^{-m} \sum_{n=0}^{\infty} \alpha^{2n} c_n,$$

where b_n and c_n are constants. Using this it is immediately obvious that the smallest power of α in $\tilde{p}_0$ will be α^0 and all other powers of α will be even. So we only need to consider the logarithmic terms; these give

$$\tilde{p}_0 = \begin{cases} \tilde{q}\left(\frac{2}{\pi}\ln(\frac{a\alpha}{2})J_m(\alpha r_0) - \frac{2}{\pi}\ln(\frac{\alpha r_0}{2})J_m(\alpha r_0)\right) J_m(\alpha r), & r < r_0 \\ \tilde{q}\left(\frac{2}{\pi}\ln(\frac{a\alpha}{2})J_m(\alpha r_0)J_m(\alpha r) - J_m(\alpha r_0)\frac{2}{\pi}\ln(\frac{\alpha r_0}{2})\right), & r > r_0 \end{cases}$$

and from this expression it is clear that all $\ln(\alpha)$ terms cancel as $\alpha \to 0$, so the singularities at $\alpha = 0$ are removable. In fact, since we showed that the powers of α are even, we find that the branch cut due to the square root in α also cancels. This means that there is no continuous spectrum and the only singularities we have are simple poles where the dispersion relation $\rho(\omega - Uk)J_m(\alpha a) = i\alpha J_m'(\alpha a)Z_{eff}$ is satisfied, which gives the discrete spectrum. Indeed, there are no singularities due to the $Y_m(\alpha r)$ and $Y_m(\alpha r_0)$ terms and so they do not contribute to the final solution.

If we now invert our local Fourier transform in η we can find the values of B_0. The η-Fourier transform may be inverted using Jordan's lemma; closing in the upper half plane when looking downstream, $x > x_0$ and $\eta > 0$, and closing in the lower half plane when looking upstream, $x < x_0$ and $\eta < 0$. The only contribution will then be from the residues of the poles inside our inversion contour. Since these poles lie on values of k that are solutions to the dispersion relation there is a one-to-one mapping from modes to poles. The residue at each pole will give us the value of B_0 for the corresponding mode.

Writing down the inversion formula for the local Fourier transforms near the source, we have the local pressure perturbation near the source in the space-time domain

$$p(t, \eta, r, \theta) = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \int_{C_\omega} \int_{C_k} \tilde{p}(\omega, k, r, m) e^{i(\omega t - k\eta - m\theta)} \, dk \, d\omega,$$

where C_k is our k Fourier inversion contour. For the ω inversion we perform a Laplace inversion with $s = i\omega$. Since we are only interested in the long time solution the only contribution from the ω inversion that we will take is the pole at ω_f, corresponding to the long time reaction to the forcing. There will also be some other ω poles in our solution,

however these will correspond to transient modes associated with the abrupt startup of our source which we will ignore here.

Once we know the locations of the modes k_* that fall inside our integration contour we can then analytically compute the residue of each mode using L'Hôpital's rule to find B_0 for that mode

$$B_0|_{k=k_*} = \pm \frac{q\mathrm{i}(\omega - Uk_*)\omega}{4} \left(J_m(\alpha r_0) \frac{\rho(\omega - Uk_*)Y_m(\alpha a) - \mathrm{i}\alpha Y_m'(\alpha a)Z_{eff}}{\frac{\partial}{\partial k}(\rho(\omega - Uk)J_m(\alpha a) - \mathrm{i}\alpha J_m'(\alpha a)Z_{eff})|_{k=k_*}} \right),$$

with $B_0 = 0$ if the mode is outside the integration contour. The $-$ sign is taken for downstream modes as the k-inversion contour is clockwise and the $+$ sign for upstream modes as the k-inversion contour is counter-clockwise.

Now that we have both slowly varying functions and have used the local solution to determine the appropriate constants for each mode, we are almost ready to calculate the global slowly varying long term solution. The only thing remaining is to determine whether our modes correspond to upstream or downstream modes, i.e. whether they fall inside or outside our η Fourier inversion contour. To determine this we use the Briggs-Bers method, which is covered in the next section.

4.8   Briggs-Bers

To maintain causality we take our transform from t to ω to be a Laplace transform with $s = i\omega$ so that there is no response before the initial time $t = 0$. For the Laplace inversion we then need to take our inversion contour to be to the right of all poles in the s-plane which corresponds to a contour below all poles in the ω-plane. This means that we are integrating along $\omega = r - \mathrm{i}R$ with R sufficiently large.

For $t < 0$ we will use Jordan's lemma and close our ω contour in the lower half plane. The k-inversion contour is then closed in the upper half plane for $x < x_0$ and the lower half plane for $x > x_0$. For there to be no poles within the ω contour we take all poles in the upper half plane to be upstream modes, $k_* \in \mathcal{K}_+$ and all poles in the lower half plane to be downstream modes $k_* \in \mathcal{K}_-$. Since there are no poles inside this contour by construction there is no pressure pertubation, as required. For $t > 0$ we now use Jordan's lemma to close the ω contour in the upper half plane and we will get a contribution from the residue of the poles inside the contour. For the k-inversion we have the same sets of upstream and downstream poles $\mathcal{K}_\pm$.

However, we are only interested in the long term solution so we only want to evaluate the contribution from the pole at ω_f. To evaluate this contribution we must raise our ω

integration contour to lie on the real axis in ω-space. Since this is a smooth deformation of the contour we expect the k Fourier inversion to also vary smoothly as the imaginary part of ω is increased. However if any modes cross the real k axis and we do not include them in the inversion for the half plane from which they originated, then the k-inversion will not vary smoothly. This means that we must track the modes $k_*(\omega)$ as $\omega \to \omega_f$, so that the modes stay in their respective sets; $\mathcal{K}_+$ or $\mathcal{K}_-$. Any mode that crosses the real k axis corresponds to a convective instability as it will be a streamwise exponentially growing mode up or downstream.

In general, when evaluating the time domain long time pressure perturbation, instead of finding our modes for ω with large negative imaginary part and tracking them as $\omega \to \omega_f$, we invert the process. We find the modes at $\omega = \omega_f$ and then track them as $\mathrm{Im}(\omega) \to -\infty$. This then allows us to classify the modes into upstream or downstream modes based on which half plane they end up in.

We can plot the Briggs-Bers trajectories for a given azimuthal wavenumber m. A comparison between the Myers and slowly varying dispersion relation (4.39) for $Z = 0.1 - 0.15\frac{\mathrm{i}}{\omega} + 0.05\mathrm{i}\omega$ is given in figure 4.3. While the cut-off acoustic modes (those to the left hand side of the k-plane) behave similarly for both dispersion relations, it is clear that the surface mode (the mode that begins in the upper-right quadrant) behaves quite differently. For the Myers dispersion relation the surface mode is unstable, and corresponds to a convective instability; while for the slowly varying dispersion relation the surface mode is stable.

4.9   Numerical methods

To solve for the long time pressure perturbation inside the duct we must first find the modes. To do this I begin by looking for modes of the dispersion relation (4.39) at the source location $x = x_0$, ignoring the $\mathcal{O}(\delta_L)$ correction terms (4.38). This is useful because the dispersion relation may then be written as:

$$f(\omega, k, m) = \frac{J_m(\alpha a)}{\alpha^{|m|}}(\omega - Uk)^2 - \frac{Zi\alpha J_m'(\alpha a)}{\rho \alpha^{|m|}}\left(\omega - \frac{ku_\zeta(0)(\gamma-1)T(0)\sqrt{\varepsilon_k}}{\sqrt{\mathrm{i}\omega\gamma p(0)X}}\right) = 0. \quad (4.57)$$

Note that the $\alpha^{|m|}$ terms cancel any solutions with $\alpha = 0$, since we know from the Green's function solution §4.7 that these do not contribute to the final solution. These terms also ensure that there is no branch cut due to the square root inside the α term as we are left with only the even powers of α. This then means that $f(\omega, k, m)$ is an analytic function of k and has no singularities. This is helpful as it means that we can use a zero counting integral to count the number of zeros, N_0, for fixed ω and m within a given integration contour, Γ.
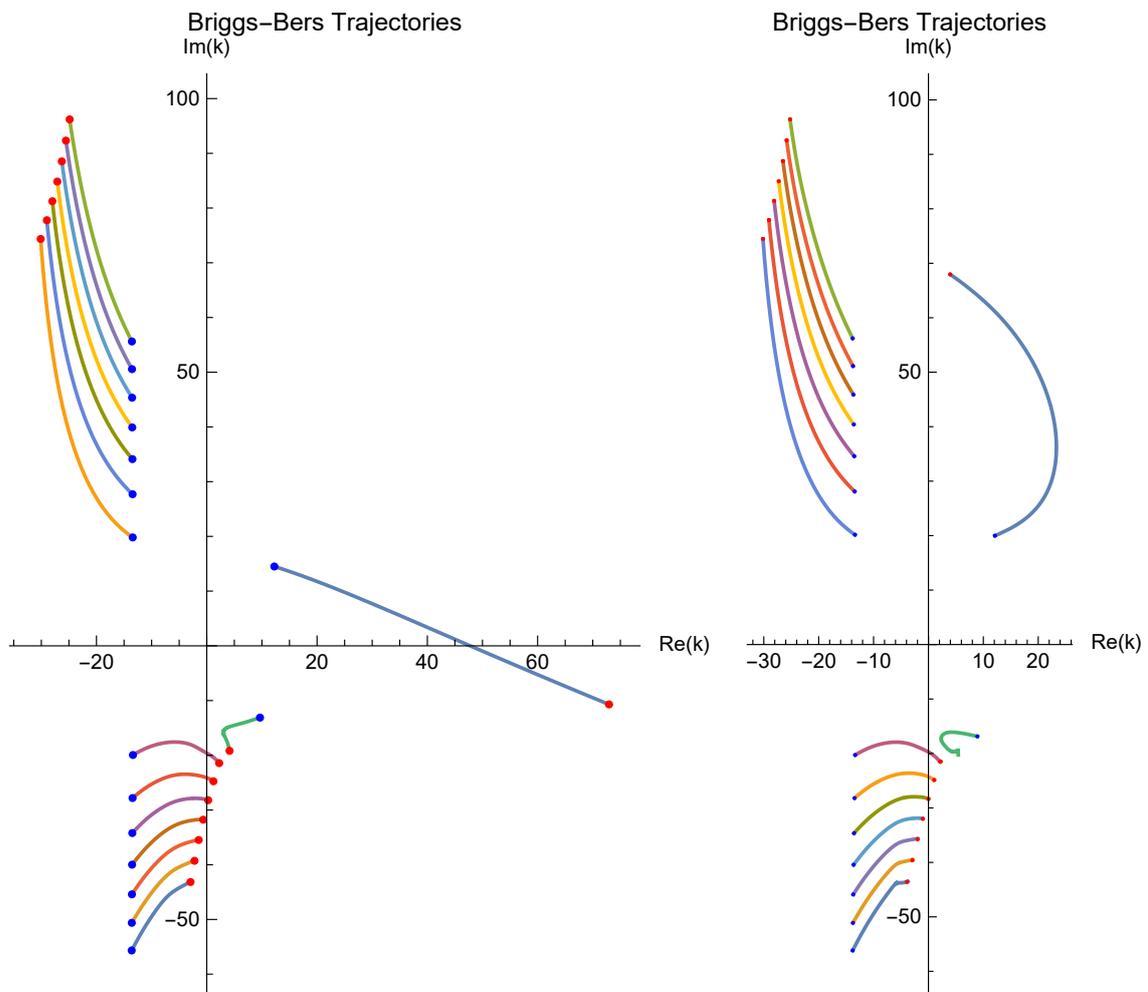
Fig. 4.3 Plot of thes Briggs-Bers trajectories for the Myers (left) and slowly varying (right) dispersion relations for $m = 15$ with $Z = 0.1 - 0.15i/\omega + 0.05i\omega$, $M = 0.7$, $a = 1$ and $\delta = 10^{-3}$. The initial positions of the modes for $\omega = 10$ are given by the blue points while the final positions for $\omega = 10 - 20i$ are given by the red points.
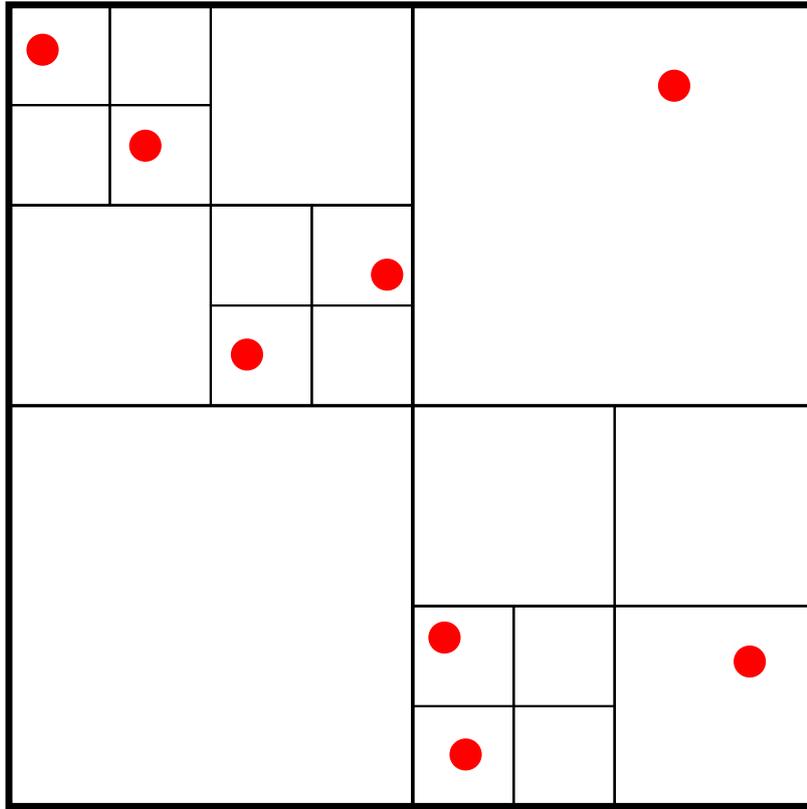
Fig. 4.4 Diagram of the subdivision process used to find the roots of the dispersion relation

Given that there are no poles inside Γ, the counting integral has the following form

$$N_0 = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(k)}{f(k)}\, dk. \tag{4.58}$$

To find the modes I then fix $\omega = \omega_f$ (as we are only interested in the long time solution) and for each m within some finite range I search for the modes $k(\omega, m)$. To do this I choose a starting contour Γ_0 to be a square of side length R and calculate the counting integral. I then subdivide to get four squares of side length $R/2$ and again count the modes within each subdivided square. This process continues until there are either no zeros within a given square or only one mode. At this point a simple root finding algorthim (e.g. Newton-Raphson) can be used to find the single zero within a square using the centre of the square as the starting value. If the root finding algorithm fails to find the mode within the square, it may be further subdivided to refine the starting value given to the root finding algorithm. A diagram of this subdivision process is shown in figure 4.4.

It should be noted that the truncation of the search range for each given m to a square of side length R is easily justified, since for sufficiently large[3] R we will only be neglecting modes with large imaginary parts that rapidly decay away from the source, and so this will only produce errors localised near to the source. Similarly the truncation of the range of m can be justified provided that we take sufficiently many m that the neglected terms are only rapidly decaying modes; typically $m \approx 10 - 20$ is sufficient.

Once these modes are found I can then find the $\mathscr{O}(\delta_L)$ correction by simply solving for the full dispersion relation with a root finding algorithm using the previously found approximate modes as initial values and sub-stepping if the difference between the old and new modes is beyond a specified tolerance.

Given the modes at $x = x_0$ we can then evaluate the constants B_0 for each mode. We then use the chain rule on f to write

$$\frac{\partial k}{\partial \omega} = -\frac{\frac{\partial f}{\partial \omega}}{\frac{\partial f}{\partial k}}. \tag{4.59}$$

This lets us solve a simple ODE to track the mode for the Briggs-Bers procedure, using the value we have for the mode at ω_f as our initial value. Once this is done the mode can be classified as either upstream or downstream. We can then use the same method to track the modes as X varies and evaluate all necessary streamwise integrals for the slow variation terms.

Once we have calculated the streamwise varying terms for each mode we can plot the long time pressure perturbation by summing over the contribution from all modes $\{k_{\omega_f,m}\}$. This procedure can be sped up computationally by pre-computing the $r - x$ terms, t terms and θ terms separately and then combining them within the sum.

4.10 Results

In this section I will show some results for the acoustics in the duct due to a point source. I will first consider the simpler case of a cylindrical duct then move on to investigate the case of a duct of varying radius.

4.10.1 Cylindrical duct

In the case of a cylindrical duct, i.e. one with $a'(x) = 0$, the solution simplifies greatly. As the outer mean flow terms no longer depend on X the only term in the dispersion relation (4.39) that is dependent on X is of order $\mathscr{O}(\sqrt{\varepsilon_k})$, and so to leading order k will be independent of

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[3]this turns out to be R greater than the roots of α^2

X as well. The result of this is that the slowly varying terms simplify. Assuming we are using the compressible Blasius boundary layer we have $\tilde{S}_{geom} = S_{bl} = 0$, $S_{meanflow}$ is a constant in *X* and $S_{viscous} \propto \frac{1}{X}$. This means that at leading order we can analytically evalute $B(X)$ to get

$$B(X) = B_0 \left(\frac{X}{X_0}\right)^{n(\omega,k)}, \tag{4.60}$$

with

$$n(\omega,k) = -\frac{\rho(\omega - Uk)S_{viscous}|_{X=1}}{S_{meanflow}Z_{\text{eff}}\omega + 2\rho U\omega}.$$

From this we can see that as expected the assumption that $B(X)$ varies slowly will break down at the leading edge, however it will hold elsewhere.

Figure 4.5 shows the pressure perturbation of the linear acoustics in a cylindrical duct due to a source at $r = 0.8$ and $x = 10.1$ with frequency $\omega_f = 10$, comparing between the Myers boundary condition and the slowly varying boundary condition (4.39). Here I use the same impedance $Z = 0.1 - 0.15\mathrm{i}/\omega + 0.05\mathrm{i}\omega$ as was used in the Briggs-Bers plots 4.3 and so we expect the Myers solution to have a convective instability downstream and the slowly varying solution to be stable.

Since $\omega_f = 10$ and $x_0 = 10.1$ we have $\varepsilon_k \approx 0.01$ and we also have $\delta = 10^{-3}$, so both parameters are small are required. I also set the source strength $q = 10^{-7}$ so that the maximum pressure perturbation is of order 10^{-5}, and so $\varepsilon \approx 10^{-5}$ is also small and $\varepsilon \ll \delta_L$ as required for the asymptotics to be valid.

To calculate these solutions I truncated the sum over *m* at the first value *m* for which there are no 'almost cut-on' modes, i.e. the first *m* for which the decay rate of the modes is sufficiently large, in this case $m_{\max} = 15$. I also limited the search in the *k*-plane for modes to a square of side length 140 centred at the origin. These approximations only affect the accuracy of the solution very near the source as the modes that have been ignored will decay very rapidly away from the source. Although it should be noted that for the Myers boundary condition the surface mode is unstable for all *m*, so we are not capturing the higher frequency downstream instabilities.

It can be seen that the Myers solution has a convective downstream instability as expected from the Briggs-Bers trajectories, whereas this instability does not occur for the slowly varying acoustics. Interestingly, the upstream acoustics appear to be stronger than those radiated downstream.

Figure 4.6 compares the result of the slowly varying boundary condition to one of the numerical simulations with realistic values for aircraft engines suggested by Brambley
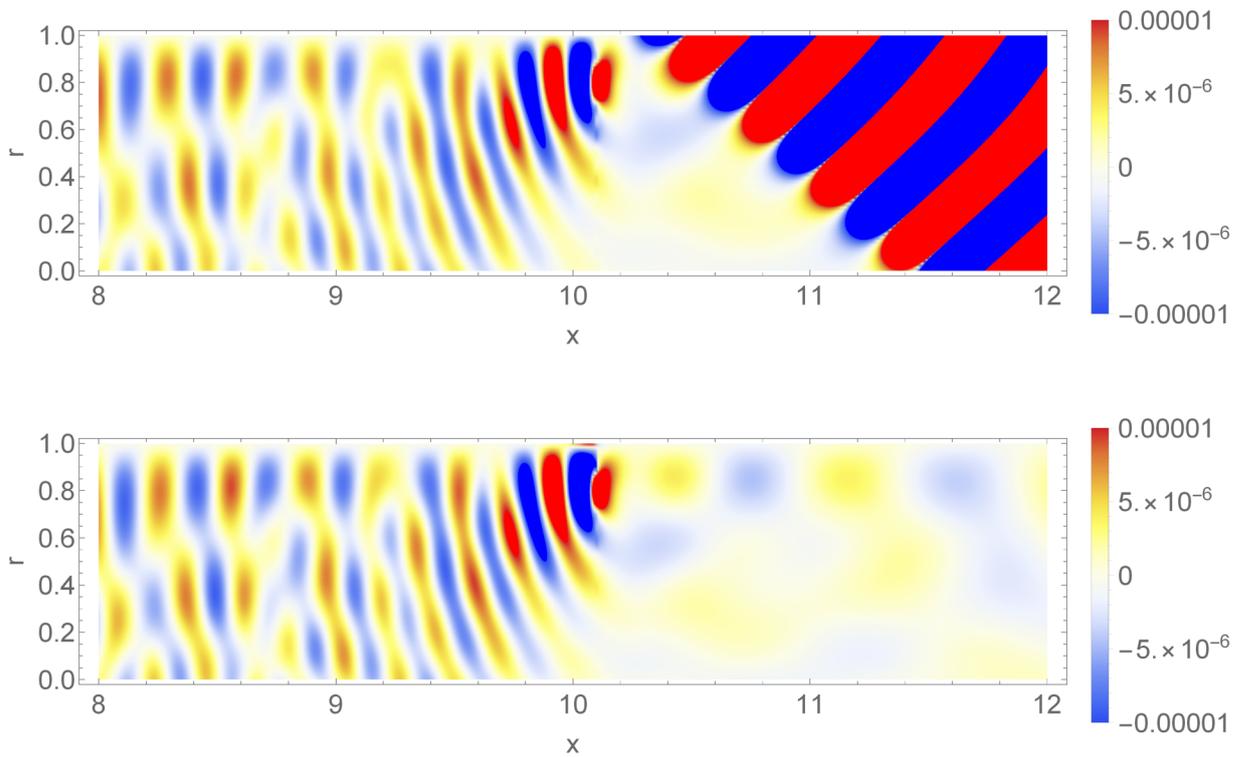
Fig. 4.5 Pressure perturbation of the linear acoustics (real part) at $\theta = 0$ due to a point source at $r = 0.8$, $\theta = 0$ and $x = 10.1$ in a cylindrical duct for the Myers (top) and slowly varying (bottom) boundary conditions for a cylindrical duct with $Z = 0.1 + 0.05i\omega - 0.15i/\omega$ for $\omega_f = 10$, and $M = 0.7$.

and Gabard [6]. This has a source with frequency $\omega_f = 31$, a lining impedance of $Z = 0.75 + 0.01\mathrm{i}\omega - 10\mathrm{i}/\omega$, $\delta = 10^{-3}$, and the mean flow Mach number is 0.5. For my solution the duct is cylindrical and the source is positioned at $r_0 = 0.7$ and $x_0 = 2.0$ so that it is 0.3 away from the wall of the duct. The transient simulations carried out in [6] used a 2D rectangular domain of height 2 and length 4 and a numerical approximation to a transient point source 0.3 away from the wall and with $x_0 = 2.0$. This means that the results may be compared even though the duct geometry is different. For my result the search window for the modes was limited to a square of side length 140 centred about the origin in k-space and the sum was truncated at $m = 8$. The source strength q was set to 2×10^{-8} which resulted in $\varepsilon \approx 10^{-5}$.

The qualitative appearance of the solutions broadly agree. However their numerical solutions appeared to show a convective instability occuring near the wall of the duct which does not appear in the slowly varying solution. However if we expand our search window for the modes and consider the Briggs-Bers trajectories, we can see that the search window was too small and we missed an unstable surface mode. From the Briggs-Bers trajectories, figure 4.7, we can see that there is indeed an unstable surface mode. This demonstrates a problem with the method of finding and tracking modes to solve for the long time pressure perturbation; it is difficult to know whether the search window for the modes is large enough to capture all potentially unstable modes. From the Briggs-Bers trajectories we can see that the growth rate of the unstable surface mode is very large and we would expect it to dominate the downstream solution. This does not occur for the numerical simulation as it is only a transient simulation rather than a long time solution.

4.10.2   Varying radius duct

For the case of a varying radius duct $B(X)$ must be integrated numerically. Here I will present the results for the linear acoustics due to a point source at $r = 0.8$, $\theta = 0$ and $x = 10.1$ for ducts of different slowly varying mean flow Mach numbers $M(X)$. The mean flow Mach numbers and radii $a(X)$ used are given in table 4.1. In both cases $|M'(x)| = 0.01$ and so we still have $\varepsilon_k \sim 0.01$. The numerical truncation used for these solutions is the same as for the parallel duct solutions above. Note that while the analytic results include corrections of order $\mathcal{O}(\varepsilon_k)$ for the plots that follow I will use the leading order non-parallel boundary layer mean flow derived in §2.4.2 for convenience.

To solve for the Myers boundary condition we now have to evaluate $k(X)$ independently at each X, since the outer mean flow velocity U now varies with X. This means that although the Myers solution does take into account some of the slowly varying nature of the mean flow
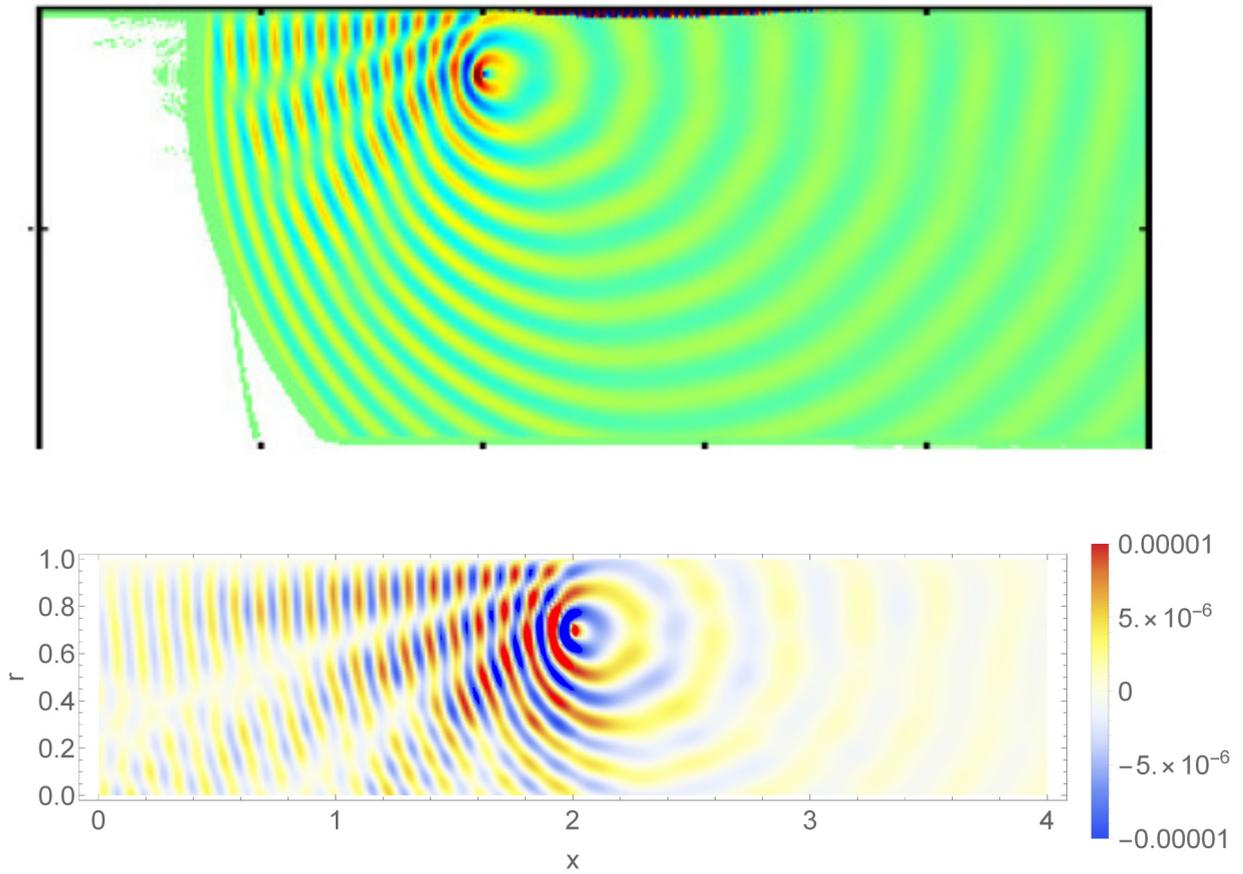
Fig. 4.6 Comparison of the numerical simulation result of Brambley and Gabard [6] at $t = 4.2$ (top) to the pressure perturbation (real part) for the slowly varying boundary condition (bottom). The white region in the numerical simulation is where the amplitude of the pressure is within the numerical precision noise
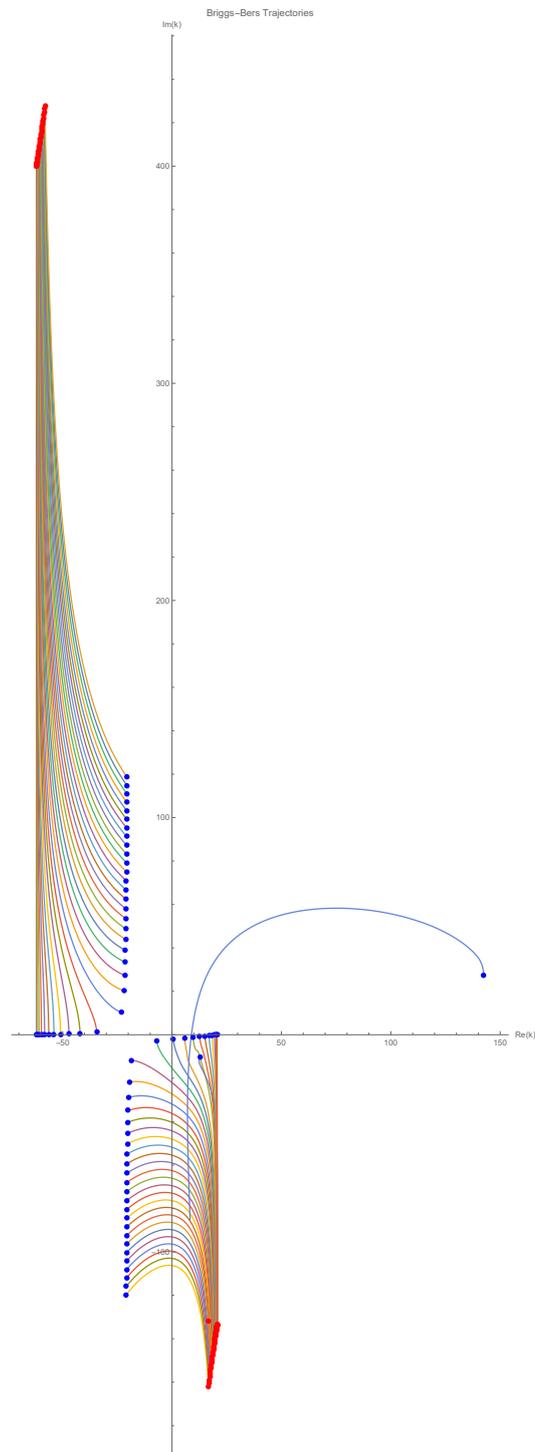
Fig. 4.7 Plot of the Briggs-Bers trajectories for the slowly varying dispersion relations for $m = 0$ with $Z = 0.75 + 0.01i\omega - 10i/\omega$, $M = 0.5$. The initial positions of the modes for $\omega = 310$ are given by the blue points while the final positions for $\omega = 31 - 200i$ are given by the red points. The unstable surface mode can be seen to the right of the plot.

it is still only a local condition, as $\mathrm{e}^{-\mathrm{i}k(X)x}$ only depends on the current location, whereas the slowly varying solution $\mathrm{e}^{-\mathrm{i}\int k(X)\,\mathrm{d}x}$ gives a global solution.

Figure 4.8 shows the long time pressure perturbation for $M = 0.7 - 0.01$. This case corresponds to a slowly expanding duct and we can see again that the Myers solution is unstable while the slowly varying solution is not, and so the solutions are very similar to the parallel duct case above.

For the accelerating flow, narrowing duct case, with $M = 0.7 + 0.01$ shown in figure 4.9, we can see that both solutions have convective instabilities downstream. These instabilities are both caused by the surface mode crossing the $\mathrm{Re}(k)$ axis and becoming a convective instability, as can be seen in the Briggs-Bers plot in table 4.1. However the growth rate of these instabilities is quite different as the location of the surface mode is quite different in each case.

We can also see that the slowly varying solution appears to be unstable upstream. Even though all the upstream modes have a positive decay rate, the solution grows because some of the upstream streamwise wavenumbers k have very slow decays rates, and for these modes the slowly varying term $B(X)$ grows. The result is that the total amplitude of the solution grows slowly upstream.

4.11 Conclusion

In this chapter I have used a WKB approximation to develop an effective impedance boundary condition that takes into account the effects of boundary layer shear, viscosity and non-parallel effects. I have shown that in certain cases it matches with previous work [5, 17]. While this new effective impedance boundary condition is more complex than the Myers boundary condition, involving integrals over the boundary layer mean flow profile, it takes into account the effects of shear, viscosity and non-parallel variations all of which are ignored by the Myers boundary condition.

I have also presented comparisons to the results of the Myers boundary condition and shown that in certain cases the surfaces modes of the slowly varying boundary condition can be stable even when they are unstable for the Myers boundary condition. This suggests that this new effective impedance boundary condition may not suffer from the ill-posedness that affects the Myers boundary condition. This ill-posedness is caused by the Myers boundary condition admitting unstable surface modes with a growth rate that increases with wavenumber. If these surfaces modes are stable under the slowly varying boundary condition then the surface modes will decay and this problem will not occur. However much more

$M(x)$	Plot $a(X)$	Briggs-Bers trajectories for $m = 0$
$0.7 - 0.01x$		
$0.7 + 0.01x$		
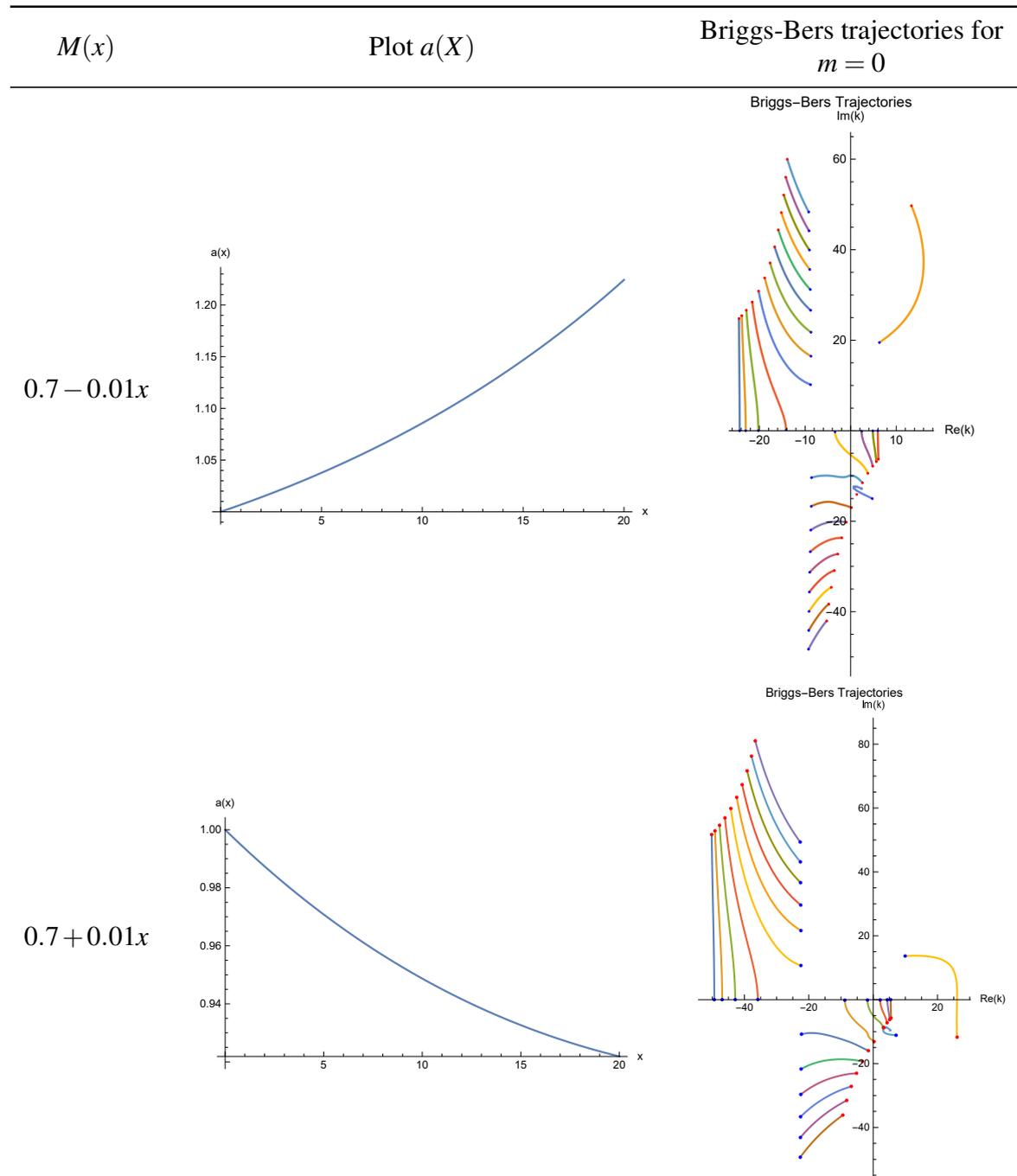
Table 4.1 Values of outer mean flow Mach number $M(x)$ used, the corresponding duct radius $a(x)$ and some of the Briggs-Bers trajectories for $m = 0$ at $x_0 = 10.1$.
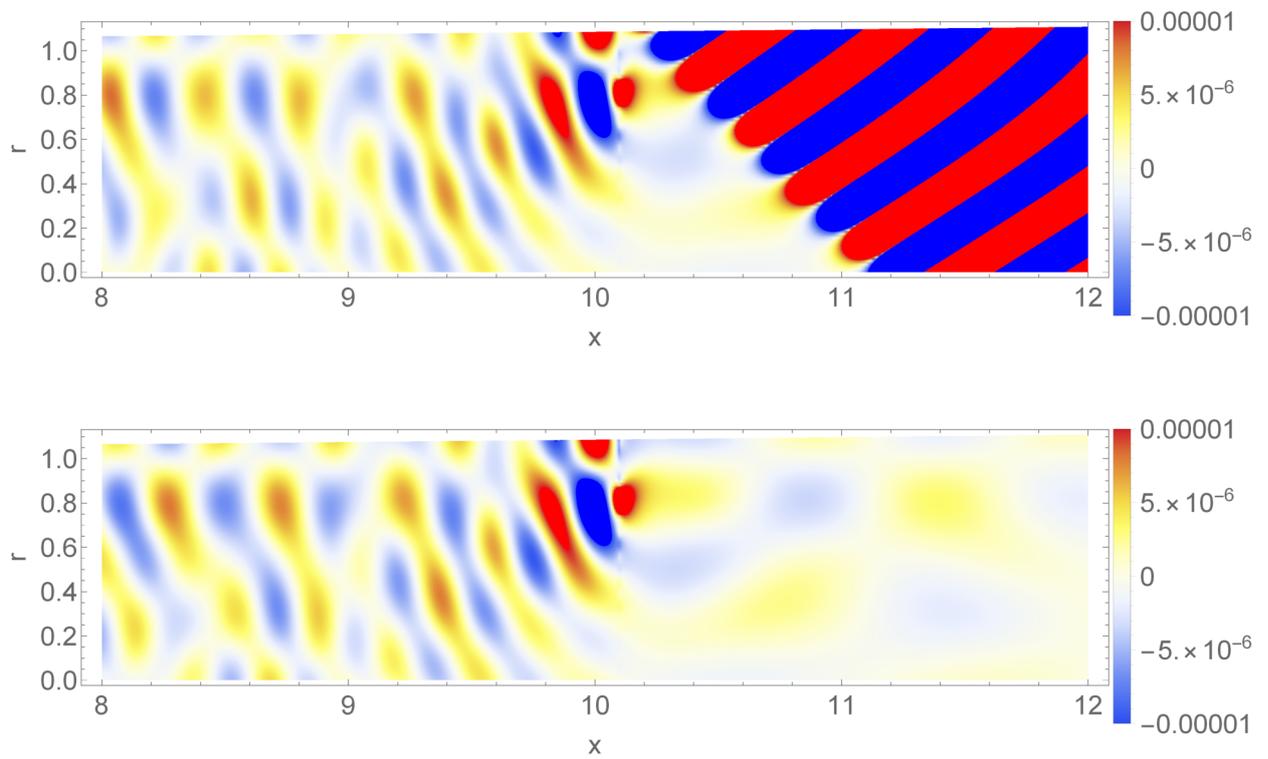
Fig. 4.8 Linear acoustics (real part) for the Myers (top) and slowly varying (bottom) boundary conditions for an expanding duct with $Z = 0.1 + 0.05i\omega - 0.15i/\omega$ for $\omega = 10$, $x = 10.1$ and $M = 0.7 - 0.01x$
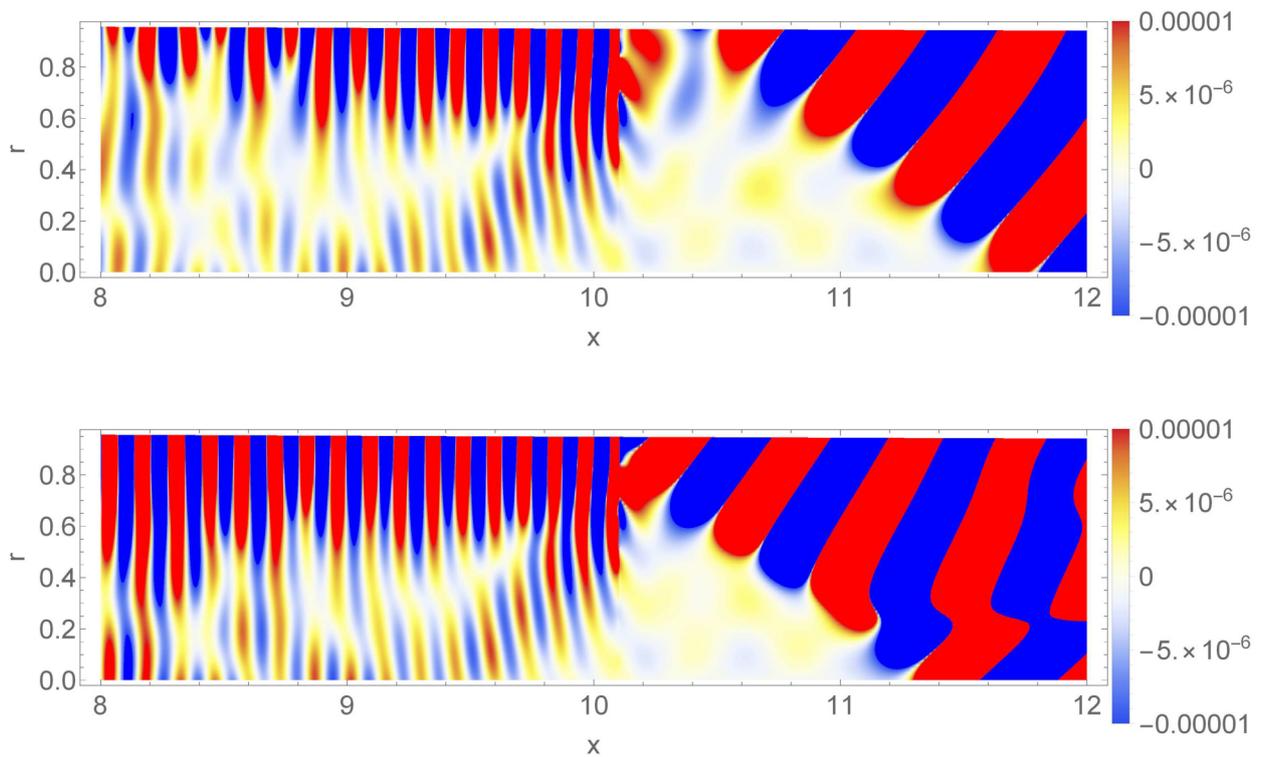
Fig. 4.9 Linear acoustics (real part) for the Myers (top) and slowly varying (bottom) boundary conditions for a narrowing duct with $Z = 0.1 + 0.05 i\omega - 0.15 i / \omega$ for $\omega = 10$, $x = 10.1$ and $M = 0.7 + 0.01x$

work would be needed to check if this behaviour is always true and to prove mathematically the well-posedness of this new boundary condition.

I have also used the WKB approximation to find the effect of a slowly varying radius duct on the acoustics. I have shown that while in some cases (fig 4.8) the slow variation appears to have relatively little effect. In other cases (fig 4.9) the slow variation of the radius can greatly affect the stability of the acoustics, both by affecting the Briggs-Bers trajectories of the modes and through the slow variation of the amplitude of the modes.

In the next chapter I will extend this work to consider the weakly nonlinear acoustics that arise in the duct due to the slowly varying linear acoustics, and investigate whether this model suffers from the same rapidly oscillating streaming solutions that were found occur in the parallel case in chapter §3.

Chapter 5

Non parallel weakly nonlinear acoustics

5.1 Introduction

In this chapter I will consider the weakly nonlinear acoustics that arise due to the slowly varying linear acoustics from chapter §4. I will use the same three layer asymptotic regime to analytically solve for the first order in ε, weakly nonlinear contributions.

Following the same scaling arguments as in §4.3 we have three different asymptotic regions. In region III, the base flow is assumed to be approximately uniform and gradients of the acoustics are assumed to be $\mathscr{O}(1)$ so that the effect of viscosity is negligible at leading order. This means that the acoustics can be treated as being in an inviscid uniform flow. In region II, the mean flow varies over a lengthscale δ_L and gradients in the acoustics are assumed to be at most $\mathscr{O}(1/\delta_L)$. Hence the effect of viscosity is still negligible at leading order and the acoustics can be treated as being in a sheared inviscid flow. In region I, the mean flow is approximately linear but gradients in the acoustics are assumed to be $\mathscr{O}(1/\lambda)$. This means that the effect of viscosity is now important. Here we treat the acoustics as being in a linearly sheared viscous flow.

In chapter §3 we found that under the parallel flow assumption some weakly nonlinear streaming modes could cause rapidly oscillating solutions to propagate out into the centre of the duct, region III, which would violate the scaling assumptions above. Here I will investigate whether it is still possible for these solutions to exist under our three-layer, slowly varying model.

5.1.1 Forcing

As in chapter §4, we will consider an oscillating point mass source at $(r_0, x_0, 0)$ which turns on at $t = 0$. This mass source has the following form (4.55)

$$q(t) = \text{Re}\left[q\delta(x - x_0)\delta(r - r_0)H(t)e^{i\omega_f t}\frac{\delta(\theta)}{r}\right],$$

where $H(t)$ is a Heaviside function, $\delta(\cdot)$ is a delta function and q is the strength of the source. Note again here that as in §4.7, there is a difference between the time varying forcing, always written $q(t)$, and the constant forcing strength q.

5.2 Linear acoustics

For the linear acoustics we will use the slowly varying solutions from chapter §4.

When we invert the ω Laplace transform we will get a contribution from the pole at ω_f and a contribution from any ω poles arising from the k-residue of B_0. These additional poles however will correspond to transient modes, so if we are interested in the long time solution we only expect to have to consider the pole at ω_f. We can then find $p_0(t, r, x, \theta)$ at long time

$$\begin{aligned}
p_0 &= \text{Re}\left[\sum_{m=-\infty}^{\infty}\sum_{\omega\in\mathcal{W}}\sum_{k_*\in\mathcal{K}_\pm}\tilde{p}_0(\omega, k, m)\right] \\
&= \text{Re}\left[\sum_{m=-\infty}^{\infty}\sum_{\omega\in\mathcal{W}}\sum_{k_*\in\mathcal{K}_\pm}B(X)J_m(\alpha r)e^{i(\omega t - \int_{x_0}^{x}k_*(X)\,dx - m\theta)}\right],
\end{aligned} \tag{5.1}$$

where $\mathcal{W}$ is the set of forcing frequencies $\{\omega_f\}$[1]. The real part here is necessary, as it was in chapter §3, as we will be taking nonlinear combinations of the leading order pressure perturbation. To evaluate the real part we take

$$\text{Re}[\tilde{p}_0] = \frac{\tilde{p}_0 + \tilde{p}_0^\star}{2}.$$

However we can note that if $\{\omega, k, m\}$ is a solution to the dispersion relation (4.39) then we can take the complex conjugate of the equation and it must still hold. However the complex conjugate of the dispersion relation is the same equation but now for $\{-\omega^\star, -k^\star, -m\}$. This means that $\{-\omega^\star, -k^\star, -m\}$ will also be a solution to the dispersion relation. Furthermore

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[1]All the following results will still hold if we do not restrict ourselves to the long time case and include the transient modes $\omega(m)$ in $\mathcal{W}$. However these modes are computationally difficult to find and there are better methods to investigate the transient stability of small perturbations.

$(B(X)_{\{\omega,k,m\}})^\star = B(X)_{\{-\omega^\star,-k^\star,-m\}}$. So we can see that

$$\tilde{p}_0(\omega,k,m)^\star = \tilde{p}_0(-\omega^\star,-k^\star,-m),$$

and so due to this symmetry of the solution we write equation (5.1) as

$$p_0 = \frac{1}{2}\sum_{\omega\in\mathcal{W}}\sum_{m=-\infty}^{\infty}\sum_{k_*\in\mathcal{K}_\pm}\tilde{p}_0(\omega,k,m) = \frac{1}{2}\sum_{\omega\in\mathcal{W}}\sum_{m=-\infty}^{\infty}\sum_{k_*\in\mathcal{K}_\pm}B(X)J_m(\alpha r)\mathrm{e}^{\mathrm{i}\left(\omega t-\int_{x_0}^x k_*(X)\,\mathrm{d}x-m\theta\right)},$$

(5.2)

where for each forcing frequency ω_f we have replaced the complex conjugate in p_0 by including $-\omega_f^\star$ in $\mathcal{W}$.

$\mathcal{K}_\pm$ corresponds to the set of upstream/downstream poles as determined by the Briggs-Bers method §4.8. That is, the set of poles which end up in the upper/lower half plane as $\mathrm{Im}(\omega_f)\to-\infty$. If $x>x_0$ we take $\mathcal{K}_-$ whereas if $x<x_0$ we take $\mathcal{K}_+$.

5.3   Weak nonlinearity

So far we have only looked at the linear acoustics that arise due to a point source. This linear solution is valid provided $\frac{\varepsilon}{\delta_L}\ll 1$ so that the nonlinear terms are much smaller than the linear acoustics. In aircraft engines the acoustics are often very loud which means that this assumption may not be true and the nonlinear acoustics may be important. I will now consider a weakly nonlinear perturbation to find a solution for the nonlinear acoustics.

For the linear acoustics we solved a system of equations of the form

$$L(p) = q(t),$$

where L is a linear operator acting on p. If we consider the nonlinear terms we now have an equation of the form

$$L(p) = Q(p,p) + q(t),$$

where Q is quadratic in p. Now using the weakly nonlinear approximation we can decompose the problem into the linear acoustics problem above and a linear problem forced by nonlinear

quantities of the linear solution

$$L(p_0) = q(t),$$
$$L(p_2) = Q(p_0, p_0).$$

We know the general form of the linear pressure perturbation p_0

$$p_0 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{\omega \in \mathcal{W}} \sum_{k(\omega,m) \in \mathcal{K}_\pm} B_k(X) J_m(\alpha_k r) e^{i(\omega t - \int_{x_0}^x k(X)\,dx - m\theta)}.$$

We can then consider quadratic quantities of p_0

$$p_0 p_0 = \frac{1}{4} \sum_{m,n=-\infty}^{\infty} \sum_{\omega, v \in \mathcal{W}} \sum_{k,l \in \mathcal{K}_\pm} B_k(X) J_m(\alpha_k r) B_l(X) J_n(\alpha_l r) e^{i(\Omega t - \int_{x_0}^x K(X)\,dx - \mathcal{M}\theta)}$$
$$= \frac{1}{4} \sum_{m,n=-\infty}^{\infty} \sum_{\omega, v \in \mathcal{W}} \sum_{k,l \in \mathcal{K}_\pm} \tilde{p}_0^a \tilde{p}_0^b,$$

where $\tilde{p}_0^a = \tilde{p}_0(\omega, k, m)$, $\tilde{p}_0^b = \tilde{p}_0(v, l, n)$, $\Omega = \omega + v$, $K(X) = k(X) + l(X)$, $\mathcal{M} = m + n$ and each sum is now a double sum over every pair of frequencies/wavenumbers (ω, v), (k, l) and (m, n). We can now write down the form of p_2

$$p_2 = \sum_{m,n=-\infty}^{\infty} \sum_{\omega, v \in \mathcal{W}} \sum_{k(\omega,m),l(v,n) \in \mathcal{K}_\pm} \tilde{p}_2(\Omega, K, \mathcal{M}) e^{i(\Omega t - \int_{x_0}^x K(X)\,dx - \mathcal{M}\theta)}, \qquad (5.3)$$

and we can solve for $\tilde{p}_2(\Omega, K, \mathcal{M})$ separately for each mode.

5.4   Leading order weakly nonlinear acoustics

For each $\omega \neq v$ or $k \neq l$ there is a pair of pairs $(\omega, k; v, l)$ and $(v, l; \omega, k)$ in the sum (5.3) that will give terms with the same frequency dependence. This mean that we can combine these contributions when solving the equations for a single mode and we just need to include a factor of $\frac{1}{2}$ to avoid double counting when we compute the double sums. This allows the solutions to be greatly simplified. I will now solve for the leading order weakly nonlinear acoustics in each of the asymptotic regions.

5.4.1 Region III

In region III, we start with the $\mathcal{O}(\varepsilon^2)$ terms of the Navier Stokes equations (2.1), ignoring the viscous terms which are assumed to be negligible in this region as discussed in §4.3. The equations for each leading order weakly nonlinear mode are then given by

$$\mathrm{i}(\Omega - UK)\tilde{\rho}_2 - \mathrm{i}K\rho\tilde{u}_2 + \frac{\rho\tilde{v}_2}{r} + \rho\tilde{v}_{2r} - \frac{\mathrm{i}\mathcal{M}\rho}{r}\tilde{w}_2 = \frac{\mathrm{i}K}{4}\tilde{\rho}_0^b\tilde{u}_0^a + \frac{\mathrm{i}\mathcal{M}}{4r}\tilde{\rho}_0^b\tilde{w}_0^a - \frac{1}{4r}(r\tilde{\rho}_0^b\tilde{v}_0^a)_r,$$
$$(5.4a)$$

$$\rho\mathrm{i}(\Omega - UK)\tilde{u}_2 - \mathrm{i}K\tilde{p}_2 = \frac{\mathrm{i}k}{4}\rho\tilde{u}_0^b\tilde{u}_0^a + \frac{\mathrm{i}m\rho}{4r}\tilde{w}_0^b\tilde{u}_0^a - \frac{\rho}{4}\tilde{v}_0^b\tilde{u}_{0r}^a - \frac{\mathrm{i}(\omega - Uk)}{4}\tilde{\rho}_0^b\tilde{u}_0^a, \qquad (5.4b)$$

$$\rho\mathrm{i}(\Omega - UK)\tilde{v}_2 + \tilde{p}_{2r} = \frac{\mathrm{i}k}{4}\rho\tilde{u}_0^b\tilde{v}_0^a + \frac{\mathrm{i}m\rho}{4r}\tilde{w}_0^b\tilde{v}_0^a - \frac{\rho}{4}\tilde{v}_0^b\tilde{v}_{0r}^a - \frac{\mathrm{i}(\omega - Uk)}{4}\tilde{\rho}_0^b\tilde{v}_0^a + \frac{\rho}{4r}\tilde{w}_0^b\tilde{w}_0^a,$$
$$(5.4c)$$

$$\rho\mathrm{i}(\Omega - UK)\tilde{w}_2 - \frac{\mathrm{i}\mathcal{M}\tilde{p}_2}{r} = \frac{\mathrm{i}k\rho}{4}\tilde{u}_0^b\tilde{w}_0^a + \frac{\mathrm{i}m\rho}{4r}\tilde{w}_0^b\tilde{w}_0^a - \frac{\rho}{4}\tilde{v}_0^b\tilde{w}_{0r}^a - \frac{\mathrm{i}(\omega - Uk)}{4}\tilde{\rho}_0^b\tilde{w}_0^a - \frac{\rho}{4r}\tilde{v}_0^b\tilde{w}_0^a,$$
$$(5.4d)$$

$$\mathrm{i}(\Omega - UK)(\rho\tilde{T}_2 - \tilde{p}_2) = \left(\frac{\mathrm{i}k}{4}\tilde{u}_0^b + \frac{\mathrm{i}m}{4r}\tilde{w}_0^b\right)(\rho\tilde{T}_0^a - \tilde{p}_0^a) - \frac{\tilde{v}_0^b}{4}(\rho\tilde{T}_0^a - \tilde{p}_0^a)_r - \frac{\mathrm{i}(\omega - Uk)}{4}\tilde{\rho}_0^b\tilde{T}_0^a,$$
$$(5.4e)$$

where $\tilde{p}_0^a = \tilde{p}_0(\omega, k, m) = B_k(X)J_m(\alpha_k r)\mathrm{e}^{\mathrm{i}(\omega t - \int_{x_0}^x k(X)\,\mathrm{d}x - m\theta)}$ is one of the leading order linear modes and $\tilde{p}_0^b = \tilde{p}_0(v, l, n)$ is another linear mode. Similarly for all other leading order linear terms, the 'a' terms correspond to a mode with frequency ω and wavenumbers k, m and the 'b' terms correspond to a mode with frequency v and wavenumbers l, n. All terms on the left hand side are of the same order and all terms on the right hand side are $\mathcal{O}(\varepsilon^2)$.

Now consider the expansion of the equation of state (2.1e) up to and including quadratic terms

$$1 + \varepsilon\sum_{\omega,k}\frac{\tilde{\rho}_0}{2\rho} + \varepsilon^2\sum_{\omega,k}\sum_{v,l}\frac{\tilde{\rho}_2}{\rho} = 1 + \varepsilon\sum_{\omega,k}\left[\frac{\tilde{p}_0}{2P} - \frac{\tilde{T}_0}{2\mathcal{T}}\right] + \varepsilon^2\sum_{\omega,k}\sum_{v,l}\left[\frac{\tilde{p}_2}{P} - \frac{\tilde{T}_2}{\mathcal{T}} - \frac{\tilde{p}_0^b\tilde{T}_0^a}{4P\mathcal{T}} + \frac{\tilde{T}_0^b\tilde{T}_0^a}{4\mathcal{T}^2}\right],$$

note here we again have a factor of $\frac{1}{2}$ in the linear terms so that we only have real quantities in the expression. Matching coefficients the of ε^2 we then have

$$\frac{\tilde{\rho}_2}{\rho} = \frac{\tilde{p}_2}{P} - \frac{\tilde{T}_2}{\mathcal{T}} - \frac{\tilde{p}_0^b\tilde{T}_0^a}{4P\mathcal{T}} + \frac{\tilde{T}_0^b\tilde{T}_0^a}{4\mathcal{T}^2} \qquad (5.5)$$

$$= \frac{\tilde{p}_2}{P} - \frac{\tilde{T}_2}{\mathcal{T}} - \frac{(\gamma - 1)\tilde{p}_0^b\tilde{p}_0^a}{4\gamma^2 P^2}, \qquad (5.6)$$

where we have used the leading order linear solution (4.5) to simplify the expression.

Using this result as well as the leading order linear solutions (4.5) to substitute all linear acoustic quantities for the linear acoustic pressure, the equations for the weakly nonlinear perturbations (5.4) can be simplified to

$$
i(\Omega - UK)\left(\frac{\tilde{p}_2}{P} - \frac{\tilde{T}_2}{T}\right) - iK\tilde{u}_2 + \frac{\tilde{v}_2}{r} + \tilde{v}_{2r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 = \frac{(\gamma - 1)i(\Omega - UK)\tilde{p}_0^a \tilde{p}_0^b}{4\gamma^2 P^2}
$$
$$
+ \frac{i}{4\gamma P(\omega - Uk)}\left[\tilde{p}_0^a \tilde{p}_0^b\left(\frac{Kk}{\rho} + \frac{\mathcal{M}m}{r^2 \rho}\right) - \frac{\tilde{p}_{0r}^a \tilde{p}_{0r}^b}{\rho} + \frac{\tilde{p}_0^a \tilde{p}_0^b}{\rho}\left(\frac{(\omega - Uk)^2}{(\gamma - 1)T} - k^2 - \frac{m^2}{r^2}\right)\right],
$$

$$
\rho i(\Omega - UK)\tilde{u}_2 - iK\tilde{p}_2 = \frac{ik}{4(v - Ul)(\omega - Uk)}\left[\tilde{p}_0^a \tilde{p}_0^b\left(\frac{lk}{\rho} + \frac{mn}{\rho r^2}\right) - \frac{\tilde{p}_{0r}^a \tilde{p}_{0r}^b}{\rho}\right] - \frac{ik\tilde{p}_0^a \tilde{p}_0^b}{4\gamma P},
$$

$$
\rho i(\Omega - UK)\tilde{v}_2 + \tilde{p}_{2r} = -\frac{1}{8(v - Ul)(\omega - Uk)}\left[\tilde{p}_0^b \tilde{p}_0^a\left(\frac{lk}{\rho} + \frac{mn}{\rho r^2}\right) - \frac{\tilde{p}_{0r}^b \tilde{p}_{0r}^a}{\rho}\right]_r + \frac{\tilde{p}_0^b \tilde{p}_{0r}^a}{4\gamma P},
$$

$$
\rho i(\Omega - UK)\tilde{w}_2 - \frac{i\mathcal{M}\tilde{p}_2}{r} = \frac{im}{4r(v - Ul)(\omega - Uk)}\left[\tilde{p}_0^a \tilde{p}_0^b\left(\frac{lk}{\rho} + \frac{mn}{\rho r^2}\right) - \frac{\tilde{p}_{0r}^a \tilde{p}_{0r}^b}{\rho}\right] - \frac{im\tilde{p}_0^a \tilde{p}_0^b}{4r\gamma P},
$$

$$
i(\Omega - UK)(\rho\tilde{T}_2 - \tilde{p}_2) = -i(\omega - Uk)\frac{\tilde{p}_0^a \tilde{p}_0^b}{4\gamma P},
$$

we can now use the symmetry $(\omega, k, m, \tilde{p}_0^a) \leftrightarrow (v, l, n, \tilde{p}_0^b)$ as discussed above to combine the contributions from the two terms with the same frequency dependence. Including the factor of $\frac{1}{2}$ to avoid double counting, the equations simplify to

$$
i(\Omega - UK)\left(\frac{\tilde{p}_2}{P} - \frac{\tilde{T}_2}{T}\right) - iK\tilde{u}_2 + \frac{\tilde{v}_2}{r} + \tilde{v}_{2r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 = \frac{i(\Omega - UK)S(r)}{4\gamma P} + \frac{\tilde{p}_0^a \tilde{p}_0^b}{4\gamma P^2}i(\Omega - UK),
$$
$$
\rho i(\Omega - UK)\tilde{u}_2 - iK\tilde{p}_2 = \frac{iK}{4}S(r),
$$
$$
\rho i(\Omega - UK)\tilde{v}_2 + \tilde{p}_{2r} = -\frac{1}{4}S'(r),
$$
$$
\rho i(\Omega - UK)\tilde{w}_2 - \frac{i\mathcal{M}\tilde{p}_2}{r} = \frac{i\mathcal{M}}{4r}S(r),
$$
$$
i(\Omega - UK)(\rho\tilde{T}_2 - \tilde{p}_2) = -\frac{i(\Omega - UK)\tilde{p}_0^a \tilde{p}_0^b}{8\gamma P},
$$

where

$$
S(r) = \frac{1}{2\rho(\omega - Uk)(v - Ul)}\left[\left(kl + \frac{mn}{r^2}\right)\tilde{p}_0^a \tilde{p}_0^b - \tilde{p}_{0r}^a \tilde{p}_{0r}^b\right] - \frac{p_0^a \tilde{p}_0^b}{2\gamma P},
$$

is a known function of the linear acoustics. Combining these equations we then have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \aleph^2 - \frac{\mathcal{M}^2}{r^2}\right)\left(\tilde{p}_2 + \frac{S}{4}\right) = (\Omega - UK)^2\left(\frac{S\rho}{2\gamma P} + \frac{\tilde{p}_0^a\tilde{p}_0^b\rho(\gamma+1)}{8\gamma^2 P^2}\right), \quad (5.7)$$

where $\aleph^2 = \frac{(\Omega-UK)^2}{(\gamma-1)\bar{T}} - K^2$. This is a forced Bessel's equation and similarly to the linear acoustics §4.4.1 we require that the pressure is regular at $r = 0$. So in region III the leading order weakly nonlinear solution is

$$\tilde{p}_2 = DJ_{\mathcal{M}}(\aleph r) - \frac{S(r)}{4} + S_f(r),$$
$$\tilde{v}_2 = \frac{-\aleph DJ'_{\mathcal{M}}(\aleph r) - S'_f(r)}{\rho\mathrm{i}(\Omega - UK)},$$

where $S_f(r)$ is the particular integral to the forced equation (5.7) and $D(X)$ is an unknown slowly varying function that will be found by matching between each layer and applying the impedance boundary condition at the wall of the duct. Note that we will find in section §5.4.4 that $D(X) \sim \mathcal{O}\left(\frac{\varepsilon^2}{\delta_L}\right)$, but we know that $S_f(r)$ must be order ε^2 as that is the order of the forcing in equation (5.7). This means that $S_f(r)$ will not contribute to the leading order term.

5.4.2 Region II

In region II we will have quadratic terms of the amplified $\mathcal{O}(\frac{\varepsilon}{\delta_L})$ linear acoustic quantities (4.14). We therefore expect the leading order nonlinear equations to be $\mathcal{O}\left(\frac{\varepsilon^2}{\delta_L^2}\right)$.

At leading order we obtain the following equations

$$\mathrm{i}(\Omega - uK)\tilde{\rho}_2 - \frac{(\rho\tilde{v}_2)_\zeta}{\delta_L} - \mathrm{i}K\rho\tilde{u}_2 = \frac{\mathrm{i}K}{4}\tilde{\rho}_0^b\tilde{u}_0^a + \frac{(\tilde{v}_0^a\tilde{\rho}_0^b)_\zeta}{4\delta_L}, \qquad (5.8a)$$

$$\rho\left[\mathrm{i}(\Omega - uK)\tilde{u}_2 - \frac{\tilde{v}_2 u_\zeta}{\delta_L}\right] = \rho\left[\frac{\mathrm{i}k\tilde{u}_0^b\tilde{u}_0^a}{4} + \frac{\tilde{v}_0^a\tilde{u}_{0\zeta}^b}{4\delta_L}\right] - \frac{\tilde{\rho}_0^b}{4}\left(\mathrm{i}(\omega - uk)\tilde{u}_0^a - \frac{\tilde{v}_0^a u_\zeta}{\delta_L}\right), \qquad (5.8b)$$

$$\frac{\tilde{p}_{2\zeta}}{\delta_L} = \mathcal{O}\left(\frac{\varepsilon^2\varepsilon_k}{\delta_L^2}\right), \qquad (5.8c)$$

$$\rho\left[\mathrm{i}(\Omega - uK)\tilde{w}_2\right] - \frac{\mathrm{i}\mathcal{M}}{a(X)}\tilde{p}_2 = \mathcal{O}\left(\frac{\varepsilon^2}{\delta_L}\right), \qquad (5.8d)$$

$$\rho\left[\mathrm{i}(\Omega - uK)\tilde{T}_2 - \frac{\tilde{v}_2 T_\zeta}{\delta_L}\right] = \rho\left[\frac{\mathrm{i}k\tilde{u}_0^b\tilde{T}_0^a}{4} + \frac{\tilde{v}_0^a\tilde{T}_{0\zeta}^b}{4\delta_L}\right] - \frac{\tilde{\rho}_0^b}{4}\left(\mathrm{i}(\omega - uk)\tilde{T}_0^a - \frac{\tilde{v}_0^a T_\zeta}{\delta_L}\right). \qquad (5.8e)$$

From these equations we can see that the nonlinear azimuthal velocity perturbation $\tilde{w}_2$ is order $\mathcal{O}(\frac{\varepsilon^2}{\delta_L})$ much less that $\tilde{u}_2$ and $\tilde{T}_2$ which are $\mathcal{O}(\frac{\varepsilon^2}{\delta_L^2})$. While we could solve for $\tilde{w}_2$ it is not necessary for the matching, so I will not solve for it here. Now we can also see that $p_{2\zeta} = \mathcal{O}\left(\frac{\varepsilon^2 \varepsilon_k}{\delta_L}\right)$ in this region. As we said in section §5.4.1 the leading order nonlinear pressure perturbation is $\mathcal{O}(\frac{\varepsilon^2}{\delta_L})$ and so in this region it will be constant at leading order.

We now only need to solve for $\tilde{u}_2$, $\tilde{v}_2$ and $\tilde{T}_2$. To do this we must first substitute for $\tilde{\rho}_2$ and $\tilde{\rho}_0$. We begin by considering the leading order $\mathcal{O}(\varepsilon^2/\delta_L^2)$ terms of the weakly nonlinear equation of state (5.5). As $\tilde{p}_0 \sim \mathcal{O}(\varepsilon)$ the equation simplifies to

$$\frac{\tilde{\rho}_2}{\rho} = -\frac{\tilde{T}_2}{T} + \frac{\tilde{T}_0^a \tilde{T}_0^b}{4T^2}. \tag{5.9}$$

Note that for the linear acoustics we also have equation (4.13):

$$\frac{\tilde{\rho}_0}{\rho} = -\frac{\tilde{T}_0}{T},$$

in this region. The leading order equations (5.8) then become

$$i(\Omega - uK)\tilde{T}_2 - \frac{T_\zeta \tilde{v}_2}{\delta_L} + \frac{T \tilde{v}_{2\zeta}}{\delta_L} + iKT\tilde{u}_2 = \frac{i(\Omega - uK)}{4T}\tilde{T}_0^a \tilde{T}_0^b + \frac{iK}{4}\tilde{T}_0^b \tilde{u}_0^a + \frac{(\tilde{v}_0^a \tilde{T}_0^b)_\zeta}{4\delta_L} - \frac{\tilde{v}_0^b \tilde{T}_0^a T_\zeta}{2T\delta_L},$$

$$i(\Omega - uK)\tilde{u}_2 - \frac{\tilde{v}_2 u_\zeta}{\delta_L} = \frac{ik}{4}\tilde{u}_0^b \tilde{u}_0^a + \frac{\tilde{v}_0^a \tilde{u}_{0\zeta}^b}{4\delta_L} + \frac{\tilde{T}_0^b}{4T}\left(i(\omega - uk)\tilde{u}_0^a - \frac{\tilde{v}_0^a u_\zeta}{\delta_L}\right),$$

$$i(\Omega - uK)\tilde{T}_2 - \frac{\tilde{v}_2 T_\zeta}{\delta_L} = \frac{ik}{4}\tilde{u}_0^b \tilde{T}_0^a + \frac{\tilde{v}_0^a \tilde{T}_{0\zeta}^b}{4\delta_L} + \frac{\tilde{T}_0^b}{4T}\left(i(\omega - uk)\tilde{T}_0^a - \frac{\tilde{v}_0^a T_\zeta}{\delta_L}\right).$$

When we substitute the results for the linear acoustics (4.14) and combine pairs using the $(\omega, k, m, \tilde{p}_0^a) \leftrightarrow (v, l, n, \tilde{p}_0^b)$ symmetry, this becomes

$$i(\Omega - uK)\tilde{T}_2 - \frac{T_\zeta \tilde{v}_2}{\delta_L} + \frac{T \tilde{v}_{2\zeta}}{\delta_L} + iKT\tilde{u}_2 = \frac{C^a C^b}{8\delta_L^2}(iKT_\zeta u_\zeta + i(\Omega - uK)T_{\zeta\zeta}),$$

$$i(\Omega - uK)\tilde{u}_2 - \frac{\tilde{v}_2 u_\zeta}{\delta_L} = \frac{C^a C^b}{8\delta_L^2}(iKu_\zeta^2 + i(\Omega - uK)u_{\zeta\zeta}),$$

$$i(\Omega - uK)\tilde{T}_2 - \frac{\tilde{v}_2 T_\zeta}{\delta_L} = \frac{C^a C^b}{8\delta_L^2}(iKu_\zeta T_\zeta + i(\Omega - uK)T_{\zeta\zeta}),$$

which we can solve analytically to find the leading order weakly nonlinear acoustic perturbations in this region

$$\tilde{p}_2 = const, \tag{5.10a}$$

$$\tilde{v}_2 = -\frac{iKC^aC^b}{8\delta_L}u_\zeta - \frac{E}{\delta_L}i(\Omega - uK), \tag{5.10b}$$

$$\tilde{u}_2 = \frac{C^aC^b}{8\delta_L^2}u_{\zeta\zeta} - \frac{E}{\delta_L}u_\zeta, \tag{5.10c}$$

$$\tilde{T}_2 = \frac{C^aC^b}{8\delta_L^2}T_{\zeta\zeta} - \frac{E}{\delta_L}T_\zeta, \tag{5.10d}$$

where $E(X)$ is a slowly varying term that will be matched to $D(X)$ in region III. C^a and C^b are the slowly varying function $C(X)$ (4.14) for the 'a' and 'b' modes respectively.

5.4.3 Region I

For the linear acoustics, in region I we found that $\tilde{u}_0 \sim \mathscr{O}(\frac{\varepsilon}{\delta_L})$ while $\tilde{T}_0 \sim \mathscr{O}(\frac{\varepsilon\sqrt{\varepsilon_k}}{\delta_L})$ as the solution for T_0 in region II becomes small when matched to region I. For the nonlinear acoustics however we find that the solution for $\tilde{T}_2$ in region II does not become small when matched to region I, as $T_{\zeta\zeta}(0) \neq 0$, and we therefore expect $\tilde{T}_2 \sim \mathscr{O}(\frac{\varepsilon^2}{\delta_L^2})$. However when we consider the nonlinear forcing by the leading order acoustic terms we find that the $\tilde{u}_2$ equation has forcing terms of order $\mathscr{O}(\frac{\varepsilon^2}{\sqrt{\varepsilon_k}\delta_L^2})$ so we expect $\tilde{u}_2$ to be of this order. This means that as in §4.4.3 the equation for $\tilde{T}_2$ will uncouple from the equations for $\tilde{u}_2$ and $\tilde{v}_2$. As we are only solving for the leading order weakly nonlinear acoustics we therefore do not need to consider the $\tilde{T}_2$ equation. So we have

$$iK\tilde{u}_2 + \sqrt{\frac{X}{\varepsilon_k}}\frac{\tilde{v}_{2y}}{\delta_L} = \mathscr{O}\left(\frac{\varepsilon^2}{\delta_L^2}\right), \tag{5.11}$$

$$i\left(\Omega - y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0)K\right)\tilde{u}_2 - \frac{\tilde{v}_2 u_\zeta(0)}{\delta_L} - \frac{(\gamma-1)^2 T(0)^2}{\gamma p}\tilde{u}_{2yy} = \frac{\tilde{v}_0^b \tilde{u}_{0y}^a \sqrt{X}}{4\delta_L\sqrt{\varepsilon_k}} + \frac{ik\tilde{u}_0^b \tilde{u}_0^a}{4}, \tag{5.12}$$

where $\tilde{v}_2 \sim \mathscr{O}\left(\frac{\varepsilon^2}{\delta_L}\right)$, $\tilde{u}_2 \sim \mathscr{O}\left(\frac{\varepsilon^2}{\sqrt{\varepsilon_k}\delta_L^2}\right)$ and equation (5.12) is correct to order $\mathscr{O}\left(\frac{\varepsilon^2\sqrt{\varepsilon_k}}{\delta_L}\right)$. The leading order equation for $\tilde{u}_2$ then simplifies to

$$i\Omega\tilde{u}_2 - \frac{(\gamma-1)^2 T(0)^2}{\gamma p}\tilde{u}_{2yy} = \frac{ivC^aC^b u_\zeta(0)}{4\sqrt{\varepsilon_k}\delta_L^2}\frac{\sqrt{i\omega\gamma pX}}{(\gamma-1)T(0)}\exp\left(\frac{-y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}\right). \tag{5.13}$$

Provided that $\Omega \neq 0$, this equation has two exponential solutions. To match to the solution in region II we must take the decaying exponential, this is possible for $\mathrm{Re}(\Omega) \neq 0$. The case $\mathrm{Re}(\Omega) = 0$ will be covered in section §5.5 below. We may then use the no-slip condition to set the remaining arbitrary constant. Using the $(\omega, k, m, \tilde{p}_0^a) \leftrightarrow (v, l, n, \tilde{p}_0^b)$ symmetry we then have the leading order solution for $\tilde{u}_2$

$$\tilde{u}_2 = \frac{C^a C^b u_\zeta(0)\sqrt{X\gamma p}}{8\sqrt{\varepsilon_k}\delta_L^2(\gamma-1)T(0)} \left(\sqrt{i\omega}e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} + \sqrt{iv}e^{-\frac{y\sqrt{iv\gamma p}}{(\gamma-1)T(0)}} - (\sqrt{i\omega} + \sqrt{iv})e^{-\frac{y\sqrt{i\Omega\gamma p}}{(\gamma-1)T(0)}} \right).$$

Note that the branch cuts for the square roots $\sqrt{i\omega}$, $\sqrt{iv}$ and $\sqrt{i\Omega}$ are taken such that the real part of the square root is positive and the exponential terms decay. For ω and v this detail is unimportant as we are only interested in acoustic perturbations with non-zero frequency, i.e. $\mathrm{Re}(\omega)$, $\mathrm{Re}(v) \neq 0$. However for Ω we need to consider the case $\mathrm{Re}(\Omega) = 0$ separately, which will be covered in section §5.5. We can now use equation (5.11) to solve for the leading order nonlinear acoustic normal velocity perturbation

$$\tilde{v}_2 = -\frac{iE\Omega}{\delta_L} - \frac{iKC^a C^b}{8\delta_L}u_\zeta(0)\left(1 + \frac{\sqrt{\omega}+\sqrt{v}}{\sqrt{\Omega}}e^{-\frac{y\sqrt{i\Omega\gamma p}}{(\gamma-1)T(0)}} - e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} - e^{-\frac{y\sqrt{iv\gamma p}}{(\gamma-1)T(0)}} \right).$$

Now we also need to solve for the leading order nonlinear pressure perturbation, however we find that, as in region II, we have

$$\tilde{p}_2 = const.$$

5.4.4 Matching

Now that we have the leading order solutions for both the weakly nonlinear pressure perturbation $\tilde{p}_2$ and the weakly nonlinear acoustic normal velocity perturbation $\tilde{v}_2$ in each region we can match these solutions and apply the impedance boundary condition to solve for the slowly varying function $D(X)$.

When we use the impedance boundary condition to solve for $D(X)$ we find that $\tilde{p}_2$ must be $\mathcal{O}(\varepsilon^2/\delta_L)$ to balance with $\tilde{v}_2$ which is amplified in the boundary layer to be $\mathcal{O}(\varepsilon^2/\delta_L)$ at the wall. At the wall in region I we then have

$$DJ_{\mathcal{M}}(\aleph a) = \tilde{p}_2(0) = Z\tilde{v}_2(0) = Z\left[-\frac{iE\Omega}{\delta_L} - \frac{iKC^a C^b}{8\delta_L}u_\zeta(0)\left(\frac{\sqrt{\omega}+\sqrt{v}}{\sqrt{\Omega}} - 1 \right) \right]. \quad (5.14)$$

Now matching $\tilde{v}_2$ between region II and region III gives

$$-\frac{E}{\delta_L}\text{i}(\Omega - UK) = \frac{-\aleph DJ'_{\mathcal{M}}(\aleph a) - S'_f(r)}{\rho\text{i}(\Omega - UK)}, \tag{5.15}$$

and since $S_f(r) \sim \mathcal{O}(\varepsilon^2)$ is of lower order this lets us substitute $E(X)$ for $D(X)$ in equation (5.14) to give an equation for $D(X)$

$$D\left(J_{\mathcal{M}}(\aleph a) - \Omega Z(\Omega)\frac{\text{i}\aleph J'_{\mathcal{M}}(\aleph a)}{\rho(\Omega - UK)^2}\right) = \frac{C^a C^b}{8\delta_L}\left(\text{i}KZ(\Omega)u_\zeta(0)\left(1 - \frac{\sqrt{w} + \sqrt{v}}{\sqrt{\Omega}}\right)\right). \tag{5.16}$$

So we find that the nonlinear pressure perturbation is a factor of $1/\delta_L$ greater than would be expected, due to the amplification of terms within the boundary layer.

This solution is not valid for acoustic streaming modes which have $\text{Re}(\Omega) = 0$ as the exponential solution in region I no longer decays and we must match it to region II. Also this solution has a singularity at $\Omega = 0$. This is because the exponential solution for $\tilde{v}_2$ in region I will no longer be valid in this case. To solve for the acoustic streaming modes with $\Omega = 0$ we will have to include extra terms in the equations to regularise our solutions near $\Omega = 0$. The solution for both of these cases is described below.

5.5   Acoustic streaming $\text{Re}(\Omega) = 0$

In this section I will consider the case of acoustic streaming modes, $\text{Re}(\Omega) = 0$. If $\Omega \neq 0$ then equation (5.13) still has exponential solutions of the form

$$\tilde{u}_2 \propto \text{e}^{\pm\frac{y\sqrt{\text{i}\Omega\gamma p}}{(\gamma-1)T(0)}}. \tag{5.17}$$

Since $\text{Re}(\Omega) = 0$ these solutions oscillate but don't decay as $y \to \infty$, so we must match to an oscillatory solution in region II. This is done in section §5.6.

If $\Omega = 0$ then the above exponential solution is not valid and we must regularise the equations to be able to solve in region I. This procedure is covered in the next section.

5.5.1   Acoustic streaming with $\Omega = 0$

In this section I will consider the special case of acoustic streaming modes with $\Omega = 0$. Note that the case $K = 0$ is unlikely to happen for an absorbing liner as we cannot have a pair of cancelling real wavenumbers since all wavelengths of sound are absorbed somewhat by the liner and hence all wavenumbers must have some imaginary part. However we are likely

to have real pair of cancelling frequencies, which give $\Omega = 0$, as our forcing is typically harmonic.

When $\Omega \to 0$, $Z(\Omega) \to \infty$, as there can be no net flow through the acoustic lining. This means that when the boundary condition is applied at the wall we must have $\tilde{v}_2|_{y=0} = 0$. Our solution in region I for $\tilde{v}_2$ is singular and even if we rewrote it in terms of an integral of $\tilde{u}_2$, which now has a constant term instead of the exponential of Ω, we would find that we still can't satisfy both the boundary condition at $y = 0$ and the matching to region II. Also we have ignored a term of the form $\sqrt{\varepsilon_k} y \tilde{u}_2$ at leading order which will become large as $y \to \infty$ in this case.

To resolve this inconsistency we must now consider both leading order and next order $\mathcal{O}(\sqrt{\varepsilon_k})$ terms of the $\tilde{u}_2$ equation together. To do this we use equation (5.11) to substitute all $\tilde{u}_2$ terms in equation (5.12) for $\tilde{v}_2$. This lets us write the $\tilde{u}_2$ equation as a third order differential equation for $\tilde{v}_2$

$$
\left(\sqrt{\frac{X}{\varepsilon_k}} \frac{\Omega}{K} - y u_\zeta(0) \right) \tilde{v}_{2y} + u_\zeta(0) \tilde{v}_2 - \frac{(\gamma-1)^2 T(0)^2}{\gamma p} \sqrt{\frac{X}{\varepsilon_k}} \frac{1}{iK} \tilde{v}_{2yyy} = -\frac{iKC^a C^b}{8\delta_L} u_\zeta(0)^2
$$
$$
- \frac{C^a C^b u_\zeta(0) \sqrt{iX\gamma p}}{8\sqrt{\varepsilon_k}\delta_L(\gamma-1)T(0)} \left(-i\omega\sqrt{\omega} e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} + i\omega\sqrt{-\omega} e^{-\frac{y\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}} \right) + E(y),
$$

$$(5.18)$$

which is correct to $\mathcal{O}\left(\frac{\varepsilon^2 \sqrt{\varepsilon_k}}{\delta_L^2} \right)$. Note that the leading order term on the right hand side is $\mathcal{O}(1/\sqrt{\varepsilon_k})$ and we have written $v = -\omega$ as we are interested in the case $\Omega = 0$. The $E(y)$ term contains subdominant $\mathcal{O}\left(\frac{\varepsilon^2}{\delta_L^2} \right)$ decaying exponential terms from the linear solutions in region I. These terms only contribute exponential terms that are a factor of $\sqrt{\varepsilon_k}$ smaller than the leading order terms and so they may safely be ignored at leading order.

We find that the dominant contribution to the particular solution comes from the first two terms. So we get

$$
\tilde{v}_2 = -\frac{iKC^a C^b}{8\delta_L} u_\zeta(0) \left(1 - e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} - e^{-\frac{y\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}} + \tilde{v}_c \right) + \mathcal{O}(\sqrt{\varepsilon_k}),
$$

where $\tilde{v}_c$ is the solution to the homogeneous equation with initial conditions $\tilde{v}_c(0) = 1$ and $\tilde{v}_c'(0) = -\frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}$. These initial conditions are required so that $\tilde{v}_2$ and $\tilde{u}_2$ are zero at the wall to satisfy the no-slip condition and impedance condition for $\Omega = 0$.

When $\Omega \neq 0$ the first term in equation (5.18) balances the $\tilde{v}_{2yyy}$ term. However for $\Omega = 0$ the remaining terms on the left hand side of (5.18) must now balance the $\tilde{v}_{2yyy}$ term. The homogeneous equation now has a simliar form to Airy's equation, due to the $y\tilde{v}_{2y}$ term and

may be written as

$$-y\tilde{v}_{cy} + \tilde{v}_c - \frac{\tilde{v}_{cyyy}}{\beta} = 0, \tag{5.19}$$

where β is a constant

$$\beta = \sqrt{\frac{\varepsilon_k}{X}} \frac{iKu_\zeta(0)\gamma p}{(\gamma - 1)^2 T(0)^2}. \tag{5.20}$$

We can see that this has one solution of the form $\tilde{v}_c = a_0 y$. To solve for the other solutions we first do a substitution $\tilde{v}_c(y) = yF(y)$. The equation then becomes

$$-y^2 F' - \frac{(yF''' + 3F'')}{\beta} = 0.$$

We can now write $f(y) = F'(y)$ to get a second order differential equation for f

$$yf'' + 3f' + \beta y^2 f = 0.$$

If we now write $g(y) = y^2 f(y)$ the equation becomes

$$\frac{g''}{y} - \frac{g'}{y^2} + \beta g = 0,$$

the first two terms can then be combined to give

$$\left(\frac{g'}{y}\right)' + \beta g = 0,$$

Now finally if we write $h'(y) = g(y)$ we have

$$\left(\frac{h''}{y}\right)' + \beta h' = 0,$$

which can be integrated to give

$$h'' + \beta yh = const.$$

Since we did not specify the integration constant in $h = \int g \, dy$ we can set the constant here to zero and we have an Airy equation for $h(y)$. We can then work backwards through the

substitutions to find that $\tilde{v}_c$ has the following solution

$$\tilde{v}_c = a_0 y + a_1 y \int^y \frac{\mathrm{Ai}'(x(-\beta)^{1/3})}{x^2} \, \mathrm{d}x + a_2 y \int^y \frac{\mathrm{Bi}'(x(-\beta)^{1/3})}{x^2} \, \mathrm{d}x, \qquad (5.21)$$

Now to set the constants a_0, a_1 and a_2 we first consider the behaviour of this solution as $y \to \infty$. For large y we have the following standard asymptotic results for the Airy functions [9, Eq. 9.7.6 & 9.7.8]

$$\mathrm{Ai}'(x(-\beta)^{1/3}) \sim -\frac{(-\beta)^{1/12}x^{1/4}}{2\sqrt{\pi}} \mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}(-\beta)^{1/2}}, \qquad (5.22)$$

$$\mathrm{Bi}'(x(-\beta)^{1/3}) \sim \frac{(-\beta)^{1/12}x^{1/4}}{\sqrt{\pi}} \mathrm{e}^{\frac{2}{3}x^{\frac{3}{2}}(-\beta)^{1/2}}, \qquad (5.23)$$

which are valid in the range

$$|\arg((-\beta)^{1/3})| < \pi/3.$$

So if we take the root of $-\beta$ that is within this arc then $\mathrm{Bi}'(x)$ will grow exponentially as x increases. This means that to be able to match our solution to region II we must set $a_2 = 0$. However if $|\arg(-\beta)^{1/3}| = \pi/3$ then the $\mathrm{Bi}'(x)$ will oscillate and not decay for large x and so we will not necessarily have $a_2 = 0$. This case is covered below in section §5.5.2.

To set the other two constants we will use the boundary conditions at $y = 0$. However we first note that the integrand is singular at $x = 0$. To remedy this problem we subtract a term of the form $\mathrm{Ai}'(0)/x^2$ from the integrand and add a contribution outside the integral to cancel it so that the solutions remains unchanged. This gives

$$\tilde{v}_c = -a_1 \mathrm{Ai}'(0) + a_0 y + a_1 y \int^y \frac{\mathrm{Ai}'(x(-\beta)^{1/3}) - \mathrm{Ai}'(0)}{x^2} \, \mathrm{d}x. \qquad (5.24)$$

Now we can use the asymptotic behaviour of $\mathrm{Ai}'(x)$ near $x = 0$ [9, Eqs. 9.2.4 & 9.4.2]

$$\mathrm{Ai}'(x) = -\frac{1}{3^{1/3}\Gamma(1/3)} + \mathcal{O}(x^2).$$

This means that the integrand is now regular at zero, so we can set the lower limit of the integrand to zero and now we can easily apply the initial conditions at $y = 0$ to get

$$\tilde{v}_c = 1 - \frac{\sqrt{\mathrm{i}\omega\gamma p} + \sqrt{-\mathrm{i}\omega\gamma p}}{(\gamma - 1)T(0)}y - y \int_0^y \frac{-3^{1/3}\Gamma(1/3)\mathrm{Ai}'(x(-\beta)^{1/3}) - 1}{x^2} \, \mathrm{d}x, \qquad (5.25)$$

which then gives

$$\tilde{v}_2 = -\frac{iKC^aC^b}{8\delta_L}u_\zeta(0)\left(2 - \frac{y(\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p})}{(\gamma-1)T(0)} - y\int_0^y \frac{\frac{\text{Ai}'(x(-\beta)^{1/3})}{\text{Ai}'(0)} - 1}{x^2}\,dx\right)$$
$$+\frac{iKC^aC^b}{8\delta_L}u_\zeta(0)\left(e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} + e^{-\frac{y\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}}\right). \tag{5.26}$$

Now to match to region II we consider the large y behaviour of $\tilde{v}_2$. Using the result of equation (C.1) and that $\text{Ai}(x)$ decays exponentially for large x (5.22) we have the long time behaviour of $\tilde{v}_2$

$$\tilde{v}_2 \sim -\frac{iKC^aC^b}{8\delta_L}u_\zeta(0)\left(2 - \frac{y(\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p})}{(\gamma-1)T(0)} - 1 - y\frac{(-\beta)^{2/3}}{\text{Ai}'(0)}\int_0^\infty \text{Ai}(x(-\beta)^{1/3})\,dx\right).$$

We can now use the result [9, Eq. 9.10.11]

$$\int_0^\infty \text{Ai}(ax)\,dx = \frac{1}{3a}, \tag{5.27}$$

which is valid for $|\arg(a)| \le \frac{\pi}{3}$ and this then gives

$$\tilde{v}_2 \sim -\frac{iKC^aC^b}{8\delta_L}u_\zeta(0)\left(-y\frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)} + \frac{\Gamma(1/3)(-\beta)^{1/3}}{3^{2/3}}y + 1\right), \tag{5.28}$$

for large y. This must match to $\tilde{v}_2$ in region II (5.10b) for small ζ

$$\tilde{v}_2 = -\frac{iKC^aC^b}{8\delta_L}u_\zeta + \frac{iEuK}{\delta_L}$$
$$\sim -\frac{iKC^aC^b}{8\delta_L}u_\zeta(0) + \frac{iKE}{\delta_L}y\sqrt{\frac{\varepsilon_k}{X}}u_\zeta(0).$$

Also for $\Omega = 0$ the matching (5.15) of E with D becomes

$$D = -\frac{E}{\delta_L \aleph J'_{\mathcal{M}}(\aleph a)}\rho(UK)^2.$$

Putting this all together we get

$$D = -\frac{\rho(UK)^2}{\aleph J'_{\mathcal{M}}(\aleph a)}\frac{C^aC^b}{8\sqrt{\varepsilon_k}\delta_L}\frac{\sqrt{i\omega\gamma pX} + \sqrt{-i\omega\gamma pX}}{(\gamma-1)T(0)} + \mathcal{O}(\varepsilon_k^{-1/3}) \sim \mathcal{O}(\varepsilon_k^{-1/2}). \tag{5.29}$$

Note we could also include Ω in the equation for $\tilde{v}_c$ (5.19) by subsituting $z = y - \Omega/Ku_\zeta\sqrt{X\varepsilon_k}$. This gives a similar form for the solution $\tilde{v}_2$ and we find that the regularised solution (5.29) is valid for $\Omega \ll \varepsilon_k^{1/3}$ and that D will be at most $\mathscr{O}(1/\sqrt{\varepsilon_k})$. These results are derived in appendix §C.2.

5.5.2 Oscillating streaming solutions for $\Omega = 0$

So far for the acoustic streaming modes with $\Omega = 0$ we have assumed that $|\arg(-\beta)^{1/3}| < \pi/3$ so that the exponential terms from the asymptotic behaviour of Ai(x) decay. We will now consider the case where $|\arg(-\beta)^{1/3}| = \pi/3$ (i.e. $|\arg(-iK)| = \pi$). These exponential terms will now oscillate and must be considered when matching, similarly to what we found in §3.

In region I the Airy function of the second kind will no longer decay, so the solution can now be written in the form

$$
\tilde{v}_c = 1 - \frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}y
$$
$$
- a_1 y \int_0^y \frac{\frac{\mathrm{Ai}'(x(-\beta)^{1/3})}{\mathrm{Ai}'(0)} - 1}{x^2}\,\mathrm{d}x - (1-a_1)y \int_0^y \frac{\frac{\mathrm{Bi}'(x(-\beta)^{1/3})}{\mathrm{Bi}'(0)} - 1}{x^2}\,\mathrm{d}x,
$$
$$
\tilde{u}_2 = -\frac{C^a C^b u_\zeta(0)\sqrt{X\gamma p}}{8\sqrt{\varepsilon_k}\delta_L^2(\gamma-1)T(0)}\left(\sqrt{i\omega}(e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} - 1) + \sqrt{-i\omega}(e^{-\frac{y\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}} - 1)\right)
$$
$$
+ \frac{C^a C^b u_\zeta(0)\sqrt{X}(-\beta)^{2/3}}{8\delta_L^2\sqrt{\varepsilon_k}}\left(\frac{a_1}{\mathrm{Ai}'(0)}\int_0^y \mathrm{Ai}(x(-\beta)^{1/3})\,\mathrm{d}x + \frac{(1-a_1)}{\mathrm{Bi}'(0)}\int_0^y \mathrm{Bi}(x(-\beta)^{1/3})\,\mathrm{d}x\right),
$$

where I have used the result of (C.1) to evaluate $\tilde{u}_2$. The above solutions satisfy the boundary conditions at the wall ($\tilde{u}_2(0) = 0$ and $\tilde{v}_2(0) = 0$) but still contain an unknown constant a_1 that must be found by matching to the solutions in region II.

To conduct the matching we first need the asymptotic behaviour of both of the integrals of the Airy functions for large y so that we can evaluate the asymptotic behaviour of $\tilde{u}_2$. For this we first use the following expression for Bi(x) [9, Eq. 9.2.10]

$$
\mathrm{Bi}(x) = e^{-\frac{i\pi}{6}}\mathrm{Ai}(xe^{-\frac{2i\pi}{3}}) + e^{\frac{i\pi}{6}}\mathrm{Ai}(xe^{\frac{2i\pi}{3}}).
$$

We will also use the integral results (5.27) and [9, Eq. 9.10.11]

$$
\int_{-\infty}^0 \mathrm{Ai}(ax)\,\mathrm{d}x = \frac{2}{3a},
$$

which are both valid for $|\arg(a)| \leq \frac{\pi}{3}$. Finally we use the large y asymptotic behaviour of the following integrals [9, Eq. 9.10.4 & 9.10.6]

$$\int_x^\infty \text{Ai}(at)\,dt \sim \frac{1}{2\sqrt{\pi}x^{3/4}a^{7/4}}e^{-\frac{2}{3}(ax)^{3/2}},$$

$$\int_{-\infty}^x \text{Ai}(at)\,dt \sim \frac{1}{\sqrt{\pi}(-x)^{3/4}a^{7/4}}\cos\left(-\frac{2}{3}(-ax)^{3/2}+\frac{\pi}{4}\right),$$

with the first integral above valid for $|\arg(a)| \leq \frac{\pi}{3}$. Combining these results we can then derive the following property for the asymptotic behaviour of the Airy function integrals for the case $|\arg(-\beta)^{1/3}| = \pi/3$

$$\int_0^y \text{Ai}(x(-\beta)^{1/3})\,dx \sim \frac{1}{3(-\beta)^{1/3}} - \frac{e^{-\frac{2}{3}y^{3/2}\sqrt{-\beta}}}{2\sqrt{\pi}y^{3/4}(-\beta)^{7/12}},$$

$$\int_0^y \text{Bi}(x(-\beta)^{1/3})\,dx \sim \frac{e^{\frac{2}{3}y^{3/2}\sqrt{-\beta}}}{\sqrt{\pi}y^{3/4}(-\beta)^{7/12}} \pm i\left(\frac{1}{(-\beta)^{1/3}} - \frac{e^{-\frac{2}{3}y^{3/2}\sqrt{-\beta}}}{2\sqrt{\pi}y^{3/4}(-\beta)^{7/12}}\right),$$

where the $\pm$ in the second integral corresponds to the sign of $\arg((-\beta)^{1/3}) = \pm i\pi/3$. We can then use these results to find the asymptotic behaviour of $\tilde{u}_2$ in region I for large y

$$\tilde{u}_2 \sim \frac{C^a C^b u_\zeta(0)\sqrt{X}}{8\delta_L^2\sqrt{\varepsilon_k}}\frac{(-\beta)^{1/12}}{\sqrt{\pi}y^{3/4}}\left(\frac{(1-a_1)}{\text{Bi}'(0)}e^{\frac{2\sqrt{-\beta}}{3}y^{3/2}} - e^{-\frac{2\sqrt{-\beta}}{3}y^{3/2}}\left(\frac{a_1}{2\text{Ai}'(0)} \pm i\frac{(1-a_1)}{2\text{Bi}'(0)}\right)\right),$$
$$\tag{5.30}$$

where I have ignored some constant terms and any terms of order $\mathcal{O}(1/y^{5/4})$ or less as we are only interested in how the oscillating behaviour of the exponential terms behaves when matched to the outer region. This matching is carried out in the next section.

5.6   Oscillating acoustic streaming in region II

We have shown that if $\Omega \neq 0$ and $\text{Re}(\Omega) = 0$ or $\Omega = 0$ and $|\arg(-iK)| = \pi$ the acoustic streaming solutions in region I oscillate and do not decay for large y. When we match these solutions to region II we will have gradients of order $\mathcal{O}\left(\frac{1}{\delta_L\sqrt{\varepsilon_k}}\right)$ which is larger than what we assumed when solving in this region. To be able to match our solutions we now consider the equations in region II for gradients $\sim 1/\sqrt{\varepsilon_k}$ to order $\mathcal{O}(\sqrt{\varepsilon_k})$. This means that we are now considering the effect of viscosity at leading order. This is the same as what was required to solve for the oscillating outer solution in the parallel flow case §3.5.5. However, here we

must first solve for the oscillating terms in region II before matching to a viscous solution in the outer region.

To solve in this region we will ignore any $\tilde{T}$ terms which oscillate with a different frequency[2] to the $\tilde{u}_2$ terms and will contribute to an additional solution for $\tilde{v}$ but not for $\tilde{u}$. Under these assumptions the leading order $\mathscr{O}\left(\frac{\varepsilon^2}{\sqrt{\varepsilon_k}\delta^2}\right)$ equations for the terms that oscillate with the same frequency as $\tilde{u}_2$ are

$$\frac{\tilde{v}_{2\zeta}}{\delta_L} + \mathrm{i}K\tilde{u}_2 = \mathscr{O}\left(\frac{\varepsilon^2}{\delta_L^2}\right), \tag{5.31a}$$

$$\mathrm{i}(\Omega - uK)\tilde{u}_2 - \frac{\tilde{v}_2 u_\zeta}{\delta_L} - \frac{\varepsilon_k(\gamma-1)^2 T^2}{X\gamma p}\left(\tilde{u}_{2\zeta\zeta} + \frac{T_\zeta}{T}\tilde{u}_{2\zeta}\right) - \tilde{u}_{2\zeta}\left(-\frac{v}{\delta_L} + \frac{\varepsilon_k u\zeta}{2X}\right) = \mathscr{O}\left(\frac{\sqrt{\varepsilon_k}\varepsilon^2}{\delta_L^2}\right). \tag{5.31b}$$

The leading order balance of equation (5.31b) is

$$\mathrm{i}(\Omega - uK)\tilde{u}_2 = \frac{\varepsilon_k(\gamma-1)^2 T^2}{X\gamma p}\tilde{u}_{2\zeta\zeta}.$$

This has exponential solutions of the form

$$\tilde{u}_2 \propto \mathrm{e}^{\pm\int^\zeta \sqrt{\frac{\mathrm{i}(\Omega - uK)X\gamma p}{\varepsilon_k(\gamma-1)^2 T^2}}\,\mathrm{d}\zeta}. \tag{5.32}$$

For $\Omega \neq 0$ it is clear that this matches to the oscillatory solutions in region I (5.17) since $u \approx 0$ for small ζ and this solution becomes

$$\tilde{u}_2 \propto \mathrm{e}^{\pm y\sqrt{\frac{\mathrm{i}\Omega\gamma p}{(\gamma-1)^2 T(0)^2}}},$$

the same as the region I solution (5.17).

For large ζ if $\mathrm{i}(\Omega - UK)$ is not purely real and negative then the argument of the exponential solution (5.32) will have a non-zero real part and so we will have an exponentially decaying solution and an exponentially growing solution. It is clear that to match to region III we must take the decaying solution and the oscillating acoustic streaming solution in region I will not propagate into the centre of the duct.

However if $\mathrm{i}(\Omega - UK)$ is purely real and negative then we will again have oscillatory solutions that don't decay. Indeed this behaviour is the same as the oscillatory solution with

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[2]The $\tilde{T}_2$ terms have exponential dependence $\propto \exp(\pm y\sqrt{\frac{\mathrm{i}\Omega\gamma p}{Pr(\gamma-1)T(0)}})$ if $\Omega \neq 0$ and $\propto \exp\pm\frac{2\sqrt{-\beta}}{3\sqrt{Pr}}y^{3/2}$if $\Omega = 0$ and so can be solved independently of the $\tilde{u}_2$ terms, however the procedure is exactly the same.

$N_\infty^2 = i(\Omega - MK)/\xi < 0$ found for the parallel mean flow in chapter §3 which in that case propagated out into the centre of the duct.

This leading order solution varies rapidly in region II and to match to region III we will also need to find the slow variation in the solution across region II. This suggests that we should try a multiple scales solution of the form

$$\tilde{v}_2 = \frac{f_\pm(\zeta)}{\delta_L} e^{\pm \int^\zeta \sqrt{\frac{i(\Omega-uK)X\gamma p}{\varepsilon_k(\gamma-1)^2 T^2}}\, d\zeta},$$

$$\tilde{u}_2 = \frac{g_\pm(\zeta)}{\sqrt{\varepsilon_k}\delta_L^2} e^{\pm \int^\zeta \sqrt{\frac{i(\Omega-uK)X\gamma p}{\varepsilon_k(\gamma-1)^2 T^2}}\, d\zeta}.$$

Solving at $\mathscr{O}\left(\frac{\varepsilon^2}{\delta_L^2}\right)$ we find the following equations for the slower varying terms $f_\pm(\zeta)$ and $g_\pm(\zeta)$

$$\frac{f_\pm}{\delta_L^2}\sqrt{\frac{i(\Omega-uK)X\gamma p}{\varepsilon_k(\gamma-1)^2 T^2}} + \frac{iKg_\pm}{\sqrt{\varepsilon_k}\delta_L^2} = 0,$$

$$-\frac{fu_\zeta}{\delta_L^2} - \frac{1}{\delta_L^2}\left(2g'_\pm - g_\pm\frac{u_\zeta K}{2(\Omega-uK)}\right)\sqrt{\frac{i(\Omega-uK)(\gamma-1)^2 T^2}{X\gamma p}}$$

$$-\frac{g_\pm}{\delta_L^2}\sqrt{\frac{i(\Omega-uK)X\gamma p}{(\gamma-1)^2 T^2}}\left(-\frac{v}{\varepsilon_k\delta_L} + \frac{u\zeta}{2X}\right) = 0.$$

These equations can then be solved to give the solutions

$$f_\pm(\zeta) = \pm\frac{\sqrt{(-iK)^2(\gamma-1)^2 T^2}g_\pm(\zeta)}{\sqrt{i(\Omega-uK)X\gamma p}}, \tag{5.33a}$$

$$g_\pm(\zeta) = \frac{g_\pm}{(\Omega-uK)^{3/4}} e^{-\int_0^\zeta \frac{X\gamma p}{2(\gamma-1)^2 T^2}\left(\frac{u\zeta}{2X} - \frac{v}{\varepsilon_k\delta_L}\right)d\zeta}$$

$$= \frac{g_\pm}{(\Omega-uK)^{3/4}} e^{-\int_0^\zeta \frac{X}{2\varepsilon_k(\gamma-1)T}\left(\frac{\varepsilon_k\rho u\zeta}{2X} - \frac{\rho v}{\delta_L}\right)d\zeta}. \tag{5.33b}$$

To check that this matches to (5.30) when $\Omega = 0$ we consider the behaviour of this solution for $\zeta \to 0$. Since $u(0) = 0$ we have

$$\tilde{u}_2 \sim \frac{g_+}{\left(y u_\zeta(0)\sqrt{\frac{\varepsilon_k}{X}}\right)^{3/4}} \mathrm{e}^{+\int_0^\zeta \sqrt{-\beta\zeta}\left(\frac{X}{\varepsilon_k}\right)^{3/4}\mathrm{d}\zeta} + \frac{g_-}{\left(y u_\zeta(0)\sqrt{\frac{\varepsilon_k}{X}}\right)^{3/4}} \mathrm{e}^{-\int_0^\zeta \sqrt{-\beta\zeta}\left(\frac{X}{\varepsilon_k}\right)^{3/4}\mathrm{d}\zeta}$$

$$\sim \frac{g_+}{\left(y u_\zeta(0)\sqrt{\frac{\varepsilon_k}{X}}\right)^{3/4}} \mathrm{e}^{\frac{2}{3}y^{3/2}\sqrt{-\beta}} + \frac{g_-}{\left(y u_\zeta(0)\sqrt{\frac{\varepsilon_k}{X}}\right)^{3/4}} \mathrm{e}^{-\frac{2}{3}y^{3/2}\sqrt{-\beta}},$$

which can be matched to (5.30) to give g_+ and g_- in terms of a_1.

Now if the exponential term in $g_\pm(\zeta)$ decays then the large oscillation will not propagate out of region II and we will not get the same behaviour as we had for the parallel flow case in chapter §3. To check the sign of the integral we begin by using the streamfunction for the boundary layer mean flow. By definition (§2.4.2) we have $\rho u = \psi_y$, $\rho v = -\psi_x - \frac{a'}{a}\psi$. This then gives the following result

$$\frac{\varepsilon_k \rho u \zeta}{2X} - \frac{\rho v}{\delta_L} = \frac{1}{\delta_L}\left[\psi \frac{a'(x)}{a(x)} + \psi_x + \psi_y \frac{y}{2x}\right].$$

The integrand in $g_\pm(\zeta)$ may be written in terms of (x, ζ) as

$$\int_0^\zeta \frac{X}{2\varepsilon_k(\gamma-1)T}\left(\frac{\varepsilon_k \rho u \zeta}{2X} - \frac{\rho v}{\delta_L}\right)\mathrm{d}\zeta = \frac{X}{2\delta_L \gamma p}\int_0^\zeta \rho\left[\psi \frac{a'(x)}{a(x)} + \psi_x\right]\mathrm{d}\zeta, \qquad (5.34)$$

where we have changed variables from (x, y) to (x, ζ) using the change of variable rules (4.11). We expect this result to be positive for all physical boundary layer flows. In general for an arbitrary boundary layer mean flow this is difficult to show but when we restrict to the case of a cylindrical duct and the compressible Blasius boundary layer mean flow (§2.4.3) we can prove the result.

For the cylindrical duct case $a'(x) = 0$ and all the outer mean flow terms are constant. In this case using the compressible Blasius boundary layer solution we know that $\psi_y = \rho u = h(\zeta)$ is[3] independent of X. Using this along with the change of variable rule (4.10) gives

$$\psi = \delta_L \int h(\zeta)\,\mathrm{d}\zeta,$$

---

[3]here $h(\zeta)$ is related to $f(\hat{\zeta})$ (2.31) but it is not necessary to know its exact form, however using the change of variable rules (4.10) it is clearly independent of X

and then we also have

$$\psi_x = \frac{\varepsilon_k}{2X}\psi,$$

since only δ_L depends on X. Using this we can now see that the integrand (5.34) has the following form

$$\frac{\varepsilon_k}{4\gamma p}\int_0^\zeta \rho\left(\int h(\zeta)\,d\zeta\right)d\zeta,$$

but since $\rho u > 0$ we have $h(\zeta) > 0$ and so the integrand is positive. This means that $g_\pm(\zeta)$ will decay exponentially in region II. For $a'(x) \neq 0$ or for different boundary layer profiles we still expect this integral to be positive, provided the choices of $a(x)$ and boundary layer profile are reasonable.

This means that the large streaming oscillatory term in $\tilde{u}_2$ decays exponentially due to the integral term in g_+ which arises only because we are including the non-parallel contributions. We can see that the first term in the integral $\frac{\varepsilon_k \rho u \zeta}{2X}$ appears from the change of variable rule (4.8) due to the fact that the mean flow boundary layer thickness δ_L depends on X and thus the mean flow is non-parallel. The second term in the integral $\frac{\rho v}{\delta_L}$ involves the normal mean flow velocity which also appears due to the fact that we have allowed the mean flow to be non-parallel, so that non-zero normal mean flow velocities are permitted. Since the large oscillatory streaming term decays in region II there is no need to consider a rapidly oscillating viscous outer solution in region III as we did in chapter §3 and the large oscillatory streaming solution will no longer propagate out into the centre of the duct.

While we found that the large oscillatory streaming solution can still appear in region I we have shown that it is confined to the boundary layer due to the interaction with the non-parallel boundary layer profile in region II. This shows that it is necessary to consider the non-parallel nature of the boundary layer when solving for the nonlinear modes and the large oscillatory behaviour in chapter §3 that extended to the centre of the duct was an artefact of the parallel flow assumption.

5.7   Results

We can now calculate the weakly nonlinear acoustics due to the leading order results presented in §4.10. Here I will present separately the time dependent nonlinear acoustics and the steady nonlinear streaming solutions.

Figure 5.1 shows the separate components of the acoustic pressure field for a source at $(r_0, x_0, \theta_0) = (0.8, 10.1, 0)$ with $\omega = 10$, $Z = 0.1 - 0.15\mathrm{i}/\omega + 0.05\mathrm{i}\omega$ and $M = 0.7$. Note that by construction, §5.3, these solutions are purely real.

It can be observed that there is more distortion at $x = x_0$, due to the truncation of the sum over the modes, than there was for the linear solutions. This is expected as any errors in the linear solutions will be magnified in the nonlinear solutions. However this distortion is confined to a fairly small region near the source and preliminary results show that this could be reduced by taking more modes, at the expense of computation time.

As before we have $\delta = 10^{-3}$ and $\varepsilon \sim 10^{-5}$. However as can be seen from the solutions the amplitude of the weakly nonlinear solutions is also of order 10^{-5} and so in this case the weakly nonlinear assumption breaks down even sooner than would be expected from the condition $\varepsilon \ll \delta_L$. This means that these solutions will only be valid for a weaker source than was used here. It should be noted that although the nonlinear acoustic pressure solution appears to have no downstream component, there are downstream nonlinear acoustics, they are just of much lower amplitude than the upstream nonlinear acoustics. Also interesting to note is that, apart from the numerical distortion near the source, the nonlinear streaming solution appears to be mostly confined to a region close to the wall of the duct with only a relatively small constant pressure contribution in the centre of the duct.

Figure 5.2 shows the nonlinear pressure perturbations corresponding to the accelerating flow, narrowing duct linear solutions of figure 4.9. Again we see that for this solution the weakly nonlinear assumption breaks down and it will only be valid for a weaker source. The downstream convective instability can clearly be seen in the nonlinear acoustics and we can see that it also causes the nonlinear streaming to fill the entire duct. However upstream the nonlinear streaming still appears to be mostly confined to a region near the wall.

5.8   Conclusion

In this chapter I have shown that the nonlinear modes that arise due to the linear acoustics are a factor of $1/\delta_L$ greater than would be expected for the nonlinear modes in a hard walled duct. This is a consequence of the amplification and subsequent interaction of certain quantities in the boundary layer. This agrees with the result from chapter §3 where we showed that the nonlinear acoustics were amplified by a factor of $\frac{1}{\delta^2}$ within the boundary layer, but only by a factor of $\frac{1}{\delta}$ in the outer region. Even when taking into account the non-parallel effects this $\mathcal{O}(\varepsilon^2/\delta_L)$ amplified nonlinear solution is still permitted in the outer region. I have also shown in this chapter that the nonlinear streaming modes are amplified further by an additional factor of $1/\sqrt{\varepsilon_k}$.
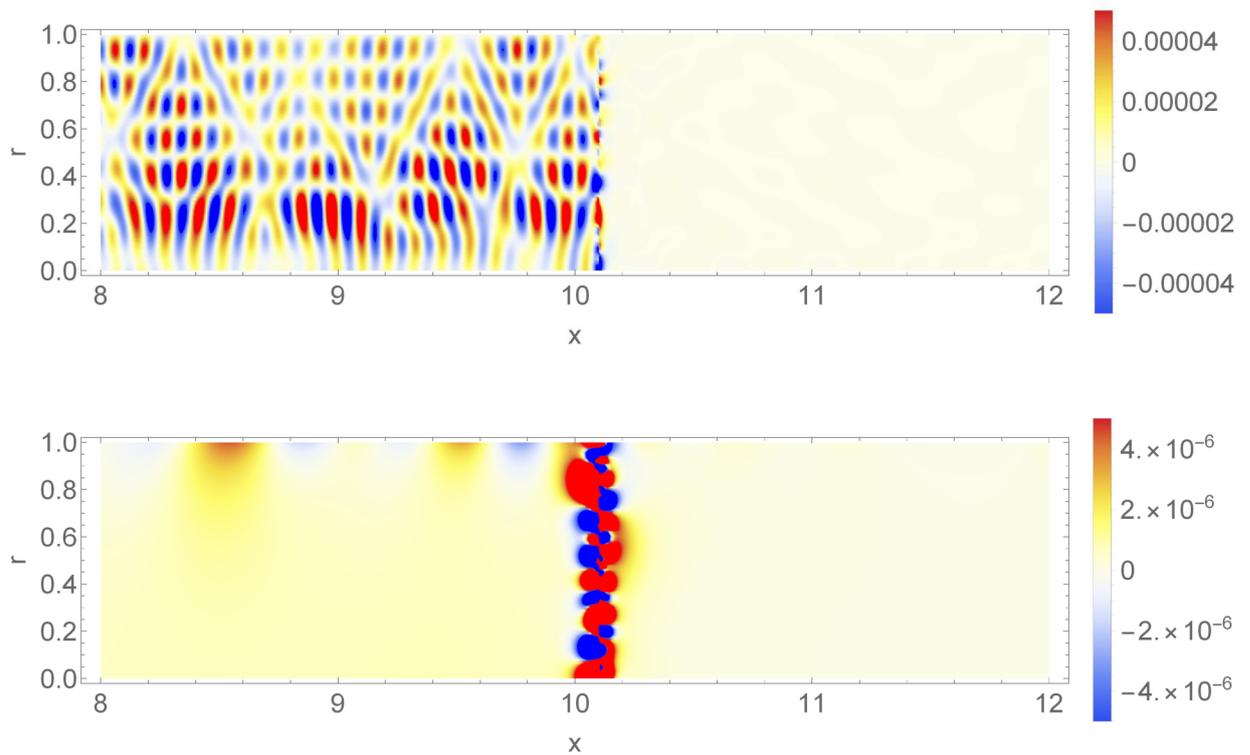
Fig. 5.1 Nonlinear acoustics (top) and nonlinear acoustic streaming (bottom) pressure perturbations in a cylindrical duct with $Z = 0.1 + 0.05\mathrm{i}\omega - 0.15\mathrm{i}/\omega$ for $\omega = 10$, $x_0 = 10.1$ and $M = 0.7$
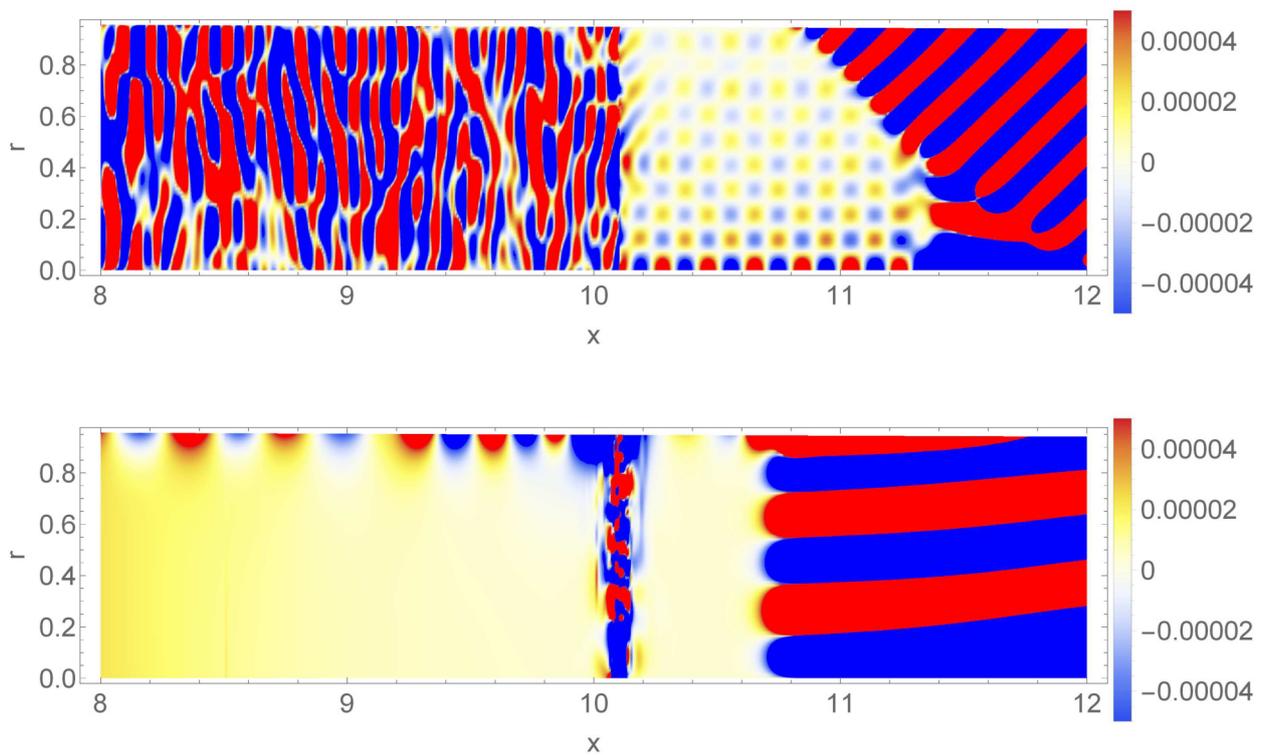
Fig. 5.2 Nonlinear acoustics for the Myers (top) and slowly varying (bottom) boundary conditions for a narrowing duct with $Z = 0.1 + 0.05\mathrm{i}\omega - 0.15\mathrm{i}/\omega$ for $\omega = 10$, $x_0 = 10.1$ and $M = 0.7 + 0.01x$

Additionally I have shown that the previously found large $\mathscr{O}(\varepsilon^2/\delta^2)$ oscillatory nonlinear solution was an artefact of the parallel flow assumption. For both the parallel §3 and non-parallel §5 solutions the same mechanism exists that can produce amplified oscillating streaming solutions in the inner region near the wall of the duct. These oscillating streaming solutions in the inner region are caused by viscosity. As the oscillating streaming solutions don't decay they must be matched to the next region away from the wall. For the parallel flow case these means that they are matched to the outer, where the effect of viscosity must now be considered, and the amplified streaming solution propagates into the centre of the duct. Whereas for the non-parallel case, these amplified streaming solutions are matched to region II, where again the effect of viscosity must now be considered. I have shown in this chapter that in region II these solutions decay exponentially away from the wall due to the interaction with non-parallel mean flow terms. These non-parallel terms are caused by the mean flow normal velocity and the streamwise variation of the mean flow boundary layer thickness.

It is expected that this exponential decay will always occur, as the only way to avoid it would be to set the mean flow shear to zero. This would either mean that there was no mean flow and so no inner region where the acoustics are amplified or would require keeping δ_L fixed and letting $x \to \infty$. However this would set $\lambda \to 0$, i.e. gives an inviscid limit, the inner-inner acoustic viscous region would become infinitely thin and the effect of viscosity would be ignored. Despite this it is possible that for certain slowly varying duct geometries or turbulent boundary layer models the decay of the amplified oscillating streaming solutions in region II could be suppressed.

Chapter 6

Conclusion

6.1 Conclusion

In this thesis I first presented the results for the weakly nonlinear acoustics inside a duct with a parallel mean flow profile, §3. This work involved a two layer matched asymptotic expansion in both ε, the acoustic perturbation amplitude and δ, the boundary layer thickness which is related to the Reynolds number by $\delta^2 = \frac{1}{Re}$. For the weakly nonlinear assumption to be valid within the inner boundary layer region I required that the assumption $\varepsilon \ll \delta \ll 1$ held. The solutions were obtained numerically in each region and the outer nonlinear acoustics were found to be amplified by a factor of $\frac{1}{\delta}$. I also showed that in certain cases a surprising, large highly oscillatory $\mathcal{O}\left(\frac{\varepsilon^2}{\delta^2}\right)$ acoustic streaming solution could propagate out into the centre of the duct.

In chapter §4 I then introduced a three-layer matched WKB solution for the linear acoustics in a duct of slowly varying radius with non-parallel mean flow. This required introducing an additional small parameter $\varepsilon_k \ll 1$ which was a measure of the streamwise variation of the mean flow. As I found that the amplification within the inner boundary layer regions still existed, I assumed $\varepsilon \ll \delta_L \ll 1$ so that the linear (or weakly nonlinear) assumption would hold, where δ_L is the streamwise varying boundary layer thickness, $\delta_L = \delta\sqrt{x}$. To avoid further assumptions that would limit the range of validity of the solutions I did not assume any relative scaling between δ_L and ε_k. While the introduction of this new small parameter ε_k meant that the accuracy of the solutions could be reduced in cases with moderate ε_k, it had the advantage that the equations could now be solved analytically in each region. Also it allowed the inner boundary layer region to be separated into two different regions; region II where the acoustics predominantly acted as if they were in an inviscid sheared flow and the inner-inner region I where viscosity acted on the acoustics. I showed that the

effective impedance condition found in this case matches to previous results found in certain limiting cases.

In chapter §5 I extended the slowly varying WKB solution to solve for the weakly nonlinear acoustics. I showed that in this case the nonlinear acoustics were still amplified by a factor of $\frac{1}{\delta_L}$. However I also showed that the previously found large amplitude, highly oscillatory, acoustic streaming solution is confined to the boundary layer in this case. This is due to the interaction of the acoustics with the non-parallel mean flow in region II.

To summarise I have shown that the nonlinear modes that arise due to the linear acoustics are a factor of $1/\delta_L$ greater than would be expected for the nonlinear modes in a hard walled duct, where δ_L is the boundary layer thickness. This is a consequence of the amplification and subsequent interaction of certain quantities within the boundary layer. I have also shown that the large $\mathcal{O}(\varepsilon^2/\delta^2)$ oscillatory nonlinear streaming solution that occured in the centre of the duct in chapter §3 is an artefact of the parallel flow assumption.

One disadvantage of the solutions presented in chapters §4 and §5 compared to the solutions in §3 is that there is now another new small parameter, ε_k. The linear solutions presented here are correct to $\mathcal{O}(\varepsilon_k)$ while the nonlinear solutions are only correct to leading order in ε_k. This means that in certain cases we may lose accuracy when compared to the parallel results in chapter §3 which only required two small parameters, δ and ε. However the solutions in chapters §4 and §5 are analytical which allows much easier computation of the pressure field and allows a more thorough investigation into the physical causes of different behaviours, whereas the solutions in §3 have to be solved numerically which means that it is much more computationally expensive to solve for the pressure field. Also, because the results of chapters §4 and §5 include non-parallel effects which have been shown to be important for the acoustic streaming modes, on balance they are more trustworthy as the scale of any neglected terms are known in terms of the small parameters.

To discuss the validity of my solutions for real world applications in aircraft engines I will now re-dimensionalise the parameters to check the range of validity.

As the fluid in aeroengines is air we have

$$c_0^* \approx 340 \text{m s}^{-1}, \quad \rho_0^* \approx 1.225 \text{kg m}^{-3}.$$

If we also assume that the radius of an aeroengine is approximately $l^* = 1\text{m}$ and the mean flow Mach number is $M = 0.5$ then, given that the kinematic viscosity of air is approximately $10^{-5}\text{m}^2\text{ s}^{-1}$, we also have that the Reynolds number is

$$Re \approx \frac{0.5 \times 340 \times 1}{10^{-5}} \approx 2 \times 10^7.$$

δ_L	Linear limit	Fully nonlinear in the boundary layer	Expected linear limit
10^{-3}	97dB	137dB	137dB–157dB
10^{-4}	77dB	117dB	137dB–157dB

Table 6.1 Table showing the limits of the linear assumption for different boundary layer thicknesses.

Also, using the values suggested by Brambley and Gabard [6] as being relevant for aero-engines we have $\omega = 31$ and $x_0 = 2.0$. This then gives the small parameters ε_k and δ_L

$$\varepsilon_k \approx 0.02, \quad \delta_L \approx 10^{-4} \text{ to } 10^{-3},$$

so while δ_L is indeed very small, ε_k is only moderately small. However as our solutions are accurate to order $\mathcal{O}(\varepsilon_k)$ we would hope to achieve a reasonably accurate result.

We can now write down the sound power level (SPL) in terms of ε,

$$SPL = 20\log_{10}\left(\frac{p^*}{2 \times 10^{-5}}\right)$$
$$= 20\log_{10}\left(\frac{\varepsilon c_0^{*2}\rho_0^*}{2 \times 10^{-5}}\right)$$
$$\approx 197 + 20\log_{10}\varepsilon.$$

Given the requirement $\varepsilon \ll \delta_L$ for the weakly nonlinear assumption to hold there is a limit to how large ε can be. It is typically assumed that nonlinearity only becomes important for $\varepsilon \gtrsim 0.001$ to 0.01, which corresponds to an SPL of 137dB–157dB. However if we assume that nonlinearity becomes important when the nondimensionalised amplitude of the acoustics in the boundary layer, $\frac{\varepsilon}{\delta_L}$, is greater than 0.01 then we will get a much lower limit where the linear assumption is valid, as shown in table 6.1. For both values of δ_L we can see that the linear assumption is not valid for an SPL greater than 100dB, far less than is normally assumed, so it is clearly important to consider the effect of nonlinearity. Also the solution becomes fully nonlinear in the boundary layer, with the acoustics the same magnitude as the mean flow, at around 120dB to 140dB. Given that aircraft engines can be louder than 140dB we can see that even the weakly nonlinear assumption may not always be valid.

6.2   Further work

For future work it would be interesting to see if a weakly nonlinear full Navier Stokes solver could be developed that takes into account the slow axial variation of the mean flow. This would allow a similar verification of the results in chapters §4 and §5 to what I was able to conduct in chapter §3.

It would also be interesting to try to match my results to the scattering solution for a cylindrical duct opening [7]. The scattering problem can be solved using the Wiener Hopf method, but an inner region near the leading edge would also be needed to match the two solutions. This would then allow the far field sound power level to be calculated analytically and could be a useful tool in analysing the effectiveness of certain acoustic linings.

Another useful extension of this work that could be made would be to derive a surface mode dispersion relation for the slowly varying effective impedance (4.40). Not only would this be interesting as it would allow a more in-depth investigation into the behaviour of the surface modes, but it could also be used to provide a starting guess for the position of the surface modes during the numerical mode search. This would mean that no, possibly unstable, surface modes would be missed, as occured in §4.10.1.

Also it would be interesting to investigate the interaction of the nonlinear effects in the boundary layer described in this thesis with the acoustic lining itself, which was treated as linear in this thesis. We know that nonlinearity within the lining can be important [32] but there may be other nonlinear effects that occur due to the interaction of the Helmholtz resonators in the liner with the nonlinear terms in region I.

Beyond this it would also be interesting to investigate whether it is possible transform the slowly varying effective impedance boundary condition to the time domain so that time dependent numerical simulations could be carried out. This would also then allow the study of the transient modes due to an acoustic source.

An attempt at experimentally validating some of the results presented here could also be interesting. Particularly whether the high aspect GFIT apparatus is indeed in the wrong parameter range to accurately study the properties of an acoustic lining for use in a moderate aspect ratio aircraft engine. Fortunately there are some future experiments planned [25] on an aeroengine that may be able to produce the results necessary to validate this result.

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Appendix A

Appendix 1

A.1 Boundary layer flow profile simplification

In this section I derive the simplification for the boundary layer flow profile equation. We begin with equation (2.26)

$$\hat{\psi}_{\hat{y}\hat{x}}\hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}}\hat{\psi}_{\hat{x}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}\hat{y}} = -\hat{\psi}_{\hat{y}}^2\frac{c_{s\hat{x}}}{c_s} - \frac{P_{\hat{x}}}{\rho c_s^2}.$$

If we use the outer mean flow equation (2.11) and use the outer speed of sound c_s to substitute for the outer density we have

$$P_{\hat{x}} = -\rho U U_{\hat{x}} = -\frac{\gamma P}{c_s^2}U U_{\hat{x}},$$

and we can also use $u = c_s\hat{\psi}_{\hat{y}}$, so that equation (2.26) can be written as

$$\hat{\psi}_{\hat{y}\hat{x}}\hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}}\hat{\psi}_{\hat{x}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}\hat{y}} = -u^2\frac{c_{s\hat{x}}}{c_s^3} + \frac{\gamma P U U_{\hat{x}}}{\rho c_s^4}. \tag{A.1}$$

Now from the outer solution we have $U = Mc_s$, and using equation (2.17) we have

$$c_{s\hat{x}} = -\frac{\frac{\gamma-1}{2}c_s M M_{\hat{x}}}{1 + \frac{\gamma-1}{2}M^2},$$

which can be substituted into equation (A.1) to give

$$\hat{\psi}_{\hat{y}\hat{x}}\hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}}\hat{\psi}_{\hat{x}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}\hat{y}} = \frac{M M_{\hat{x}}}{1 + \frac{\gamma-1}{2}M^2}\left(\frac{\gamma P}{\rho c_s^2} + \frac{u^2\frac{\gamma-1}{2}}{c_s^2}\right). \tag{A.2}$$

We can then use equation (2.17) again and the equation of state (2.1e) to simplify further giving equation (2.27)

$$\hat{\psi}_{\hat{y}\hat{x}}\hat{\psi}_{\hat{y}} - \hat{\psi}_{\hat{y}\hat{y}}\hat{\psi}_{\hat{x}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}\hat{y}} = \frac{MM_{\hat{x}}}{\frac{1}{\gamma-1} + \frac{M_0^2}{2}}\left(T + \frac{u^2}{2}\right). \tag{A.3}$$

A.2 Boundary layer temperature profile equation

In this section I derive the simplification for the boundary layer temperature profile equation. Starting with equation (2.33)

$$\hat{\psi}_{\hat{y}}\tau_{\hat{x}} - \hat{\psi}_{\hat{x}}\tau_{\hat{y}} - \frac{\delta^2}{Pr}\tau_{\hat{y}\hat{y}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}}^2 = \hat{\psi}_{\hat{y}}\frac{P_{\hat{x}}}{\rho} - 2c_s c_{s\hat{x}}\tau\hat{\psi}_{\hat{y}}, \tag{A.4}$$

we can rewrite the last term in terms of T and then use the equation of state (2.1e) to write it in terms of the pressure and density. This gives

$$\hat{\psi}_{\hat{y}}\tau_{\hat{x}} - \hat{\psi}_{\hat{x}}\tau_{\hat{y}} - \frac{\delta^2}{Pr}\tau_{\hat{y}\hat{y}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}}^2 = \frac{\hat{\psi}_{\hat{y}}}{\rho}\left(P_{\hat{x}} - \frac{2\gamma P c_{s\hat{x}}}{(\gamma-1)c_s}\right). \tag{A.5}$$

Now we use the outer mean flow solution (2.18): $P = \frac{\rho^\gamma}{\gamma}$ and the fact that the speed of sound satisfies

$$c_s^2 = \frac{\gamma P}{\rho} = \rho^{\gamma-1},$$

which then gives

$$P_{\hat{x}} - \frac{2\gamma P c_{s\hat{x}}}{(\gamma-1)c_s} = \rho^{\gamma-1}\rho_{\hat{x}} - \frac{2\rho^{\gamma\frac{(\gamma-1)}{2}}c_s\rho_{\hat{x}}}{(\gamma-1)\rho c_s} = 0,$$

and so equation (2.33) simplifies to equation (2.34)

$$\hat{\psi}_{\hat{y}}\tau_{\hat{x}} - \hat{\psi}_{\hat{x}}\tau_{\hat{y}} - \frac{\delta^2}{Pr}\tau_{\hat{y}\hat{y}} - \delta^2\hat{\psi}_{\hat{y}\hat{y}}^2 = 0. \tag{A.6}$$

Appendix B

B.1 Outer solution for self-interaction nonlinear oscillating mode

In this section I will derive the nonlinear self-interaction outer solutions in the case $N_\infty^2 < 0$. From equation (3.52) we know that for the oscillating solution both $\tilde{u}_2$ and $\tilde{T}_2$ are of order $\mathcal{O}(\varepsilon^2/\delta^2)$ while $\tilde{v}_2$ is order $\mathcal{O}(\varepsilon^2/\delta)$ and $\tilde{w}_2$ and $\tilde{p}_2$ are order $\mathcal{O}(\varepsilon^2)$. To solve for the slowly varying, non-oscillating components of $\tilde{v}_2$ and $\tilde{p}_2$ we must expand the equations to $\mathcal{O}(\varepsilon^2)$, this gives the following system of equations

$$i(\Omega - Mk)(\gamma \tilde{p}_2 - (\gamma - 1)\tilde{T}_2) - iK\tilde{u}_2 + \tilde{v}_{2r} + \frac{\tilde{v}_2}{r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 = P_{RHS}, \tag{B.1a}$$

$$i(\Omega - Mk)\tilde{u}_2 + iK\tilde{p}_2 - \frac{\delta^2 \xi}{r}(r\tilde{u}_{2r})_r + \xi\delta^2\left(2K^2\tilde{u}_2 + \frac{\mathcal{M}^2}{r^2}\tilde{u}_2 + iK\tilde{v}_{2r}\right) = U_{RHS}, \tag{B.1b}$$

$$i(\Omega - Mk)\tilde{v}_2 + \tilde{p}_{2r} + \xi\delta^2\left(-\frac{\tilde{v}_{2r}}{r} - \tilde{v}_{2rr} - \left(1 + \frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)\left(\tilde{v}_{2rr} + \frac{\tilde{v}_{2r}}{r} - iK\tilde{u}_{2r}\right)\right) = V_{RHS}, \tag{B.1c}$$

$$i(\Omega - MK)\tilde{w}_2 - \frac{i\mathcal{M}}{r}\tilde{p}_2 + \xi\delta^2\left(-\tilde{w}_{2rr} + \left(1 + \frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)(\frac{im}{r}\tilde{v}_{2r} + \frac{km}{r}\tilde{u}_2)\right) = W_{RHS}, \tag{B.1d}$$

$$i(\Omega - MK)\tilde{T}_2 - i(\Omega - MK)\tilde{p}_2 + \frac{\xi\delta^2}{Pr}\left(K^2\tilde{T}_2 + \frac{\mathcal{M}^2}{r^2}\tilde{T}_2 - \frac{1}{r}(r\tilde{T}_{2r})_r\right) = T_{RHS}, \tag{B.1e}$$

where P_{RHS}, U_{RHS}, V_{RHS}, W_{RHS} and T_{RHS} are the forcing terms that arise from terms quadratic in the linear leading order acoustics. They are given by

$$P_{RHS} = \frac{1}{4}\left(N_\infty^2 \xi (\gamma - 1)\tilde{p}_1^\star \tilde{T}_1 + \mathrm{i}K\tilde{p}_1^\star \tilde{u}_1 - \frac{1}{r}(r\tilde{p}_1^\star \tilde{v}_1)_r + \frac{\mathrm{i}\mathcal{M}}{r}\tilde{p}_1^\star \tilde{w}_1 \right),$$

$$U_{RHS} = \frac{1}{4}\left(-\eta_\infty^2 \xi \tilde{u}_1 \tilde{p}_1^\star + \mathrm{i}k\tilde{u}_1^\star \tilde{u}_1 - \tilde{v}_1^\star \tilde{u}_{1r} + \frac{\mathrm{i}m}{r}\tilde{w}_1^\star \tilde{u}_1 \right),$$

$$V_{RHS} = \frac{1}{4}\left(-\eta_\infty^2 \xi \tilde{v}_1 \tilde{p}_1^\star + \mathrm{i}k\tilde{u}_1^\star \tilde{v}_1 - \tilde{v}_1^\star \tilde{v}_{1r} + \frac{\mathrm{i}m}{r}\tilde{w}_1^\star \tilde{v}_1 + \frac{1}{r}\tilde{w}_1^\star \tilde{w}_1 \right),$$

$$W_{RHS} = \frac{1}{4}\left(-\eta_\infty^2 \xi \tilde{w}_1 \tilde{p}_1^\star + \mathrm{i}k\tilde{u}_1^\star \tilde{w}_1 - \tilde{v}_1^\star \tilde{w}_{1r} + \frac{\mathrm{i}m}{r}\tilde{w}_1^\star \tilde{w}_1 - \frac{1}{r}\tilde{v}_1^\star \tilde{w}_1 \right),$$

$$T_{RHS} = \frac{1}{4}\left(-\eta_\infty^2 \xi \tilde{T}_1 \tilde{p}_1^\star \right).$$

Expanding the streamwise momentum equation (B.1b) and using the transformation $r = R - \delta y$ as in §3.5.5 gives a set of differential equations at different orders in δ

$$\mathcal{O}\left(\frac{\varepsilon^2}{\delta^2}\right): \qquad \xi N_\infty^2 \tilde{u}_{O0} - \xi \tilde{u}_{O0yy} = 0, \tag{B.2a}$$

$$\mathcal{O}\left(\frac{\varepsilon^2}{\delta}\right): \qquad \xi N_\infty^2 \tilde{u}_{O1} - \xi \tilde{u}_{O1yy} = -\xi \left(2\tilde{u}_{O0yR} - \frac{\tilde{u}_{O0y}}{R} \right) = 0, \tag{B.2b}$$

$$\mathcal{O}(\varepsilon^2): \qquad \xi N_\infty^2 \tilde{u}_{O2} - \xi \tilde{u}_{O2yy} + \mathrm{i}K\tilde{p}_{O0}$$
$$+ \xi \left(2k^2 \tilde{u}_{O0} + \frac{\mathcal{M}^2}{r^2}\tilde{u}_{O0} - \mathrm{i}K\tilde{v}_{O0y} - \tilde{u}_{O0RR} - \frac{\tilde{u}_{O0R} - \tilde{u}_{O1y}}{R} + 2\tilde{u}_{O1Ry} \right) = U_{RHS}. \tag{B.2c}$$

Now using the results from §3.5.5 we can see that $\tilde{u}_{O0}$, $\tilde{u}_{O1}$ and $\tilde{v}_{O0y}$ only contain oscillating terms $\propto e^{\pm \mathrm{i}fy}$. So they will not affect the non-oscillating part $\hat{u}_O(R)$ of $\tilde{u}_O$ which will satisfy

$$\xi N_\infty^2 \hat{u}_O + \mathrm{i}K\hat{p}_O = U_{RHS}. \tag{B.3}$$

We use the same argument to find similar expressions for $\hat{T}_O$ and $\hat{w}_O$

$$\xi N_\infty^2 \hat{w}_O + \frac{\mathrm{i}\mathcal{M}}{r}\hat{p}_O = W_{RHS}, \tag{B.4}$$

$$\xi N_\infty^2 \hat{T}_O + \xi N_\infty^2 \hat{p}_O = T_{RHS}. \tag{B.5}$$

Now for $\tilde{v}_O$ we have

$$\mathcal{O}\left(\frac{\varepsilon^2}{\delta}\right): \qquad \tilde{v}_{O1y} + iK\tilde{u}_{O1} + i(\Omega - MK)(\gamma - 1)\tilde{T}_{O1} = \tilde{v}_{O0R} + \frac{\tilde{v}_{O0}}{R},$$

$$\mathcal{O}(\varepsilon^2): \qquad \tilde{v}_{O2y} + iK\tilde{u}_{O2} + i(\Omega - MK)(\gamma - 1)\tilde{T}_{O2} - \tilde{v}_{O1R} - \frac{\tilde{v}_{O1}}{R}$$
$$- i(\Omega - MK)\gamma\tilde{p}_{O0} + \frac{i\mathcal{M}}{r}\tilde{w}_{O0} = -P_{RHS}.$$

The first equation tells us that $\tilde{v}_{O0}$ can't have a non-oscillating part, we can then use the second equation to get an expression for the $\mathcal{O}(\varepsilon^2)$ non-oscillating part of $\tilde{v}_{O1}$

$$-iK\hat{u}_O - i(\Omega - MK)(\gamma - 1)\hat{T}_O + \hat{v}_{OR} + \frac{\hat{v}_O}{R} + i(\Omega - MK)\gamma\hat{p}_O - \frac{i\mathcal{M}}{r}\hat{w}_O = P_{RHS}.$$

Finally we consider the pressure

$$\mathcal{O}\left(\frac{\varepsilon^2}{\delta}\right): \quad \xi N_\infty^2 \tilde{v}_{O0} - \tilde{p}_{O0y} - iK\xi\tilde{u}_{O0y} - 2\xi\tilde{v}_{O0yy} - \xi\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)(iK\tilde{u}_{O0y} + \tilde{v}_{O0yy}) = 0,$$

this gives the oscillating part of the pressure

$$\tilde{p}_{O0} = \delta^2\xi^2 N_\infty^2(\gamma - 1)\left(2 + \frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3} - \frac{1}{Pr}\right)\tilde{T}_{O0} + \hat{p}_O(R).$$

We can then take the non-oscillating parts of the $\mathcal{O}(\varepsilon^2)$ equation to find an expression for $\hat{p}_O$

$$i(\Omega - MK)\hat{v}_O + \hat{p}_{Or} = V_{RHS}. \tag{B.6}$$

We can now combine the equations for $\hat{u}_O$, $\hat{v}_O$, $\hat{w}_O$, $\hat{T}_O$ and $\hat{p}_O$ to get an equation for $\hat{p}_O(R)$ which we find is the same equation as for the non-oscillating case (3.52). $\hat{v}_O$ is then given by (B.6).

B.2   Derivation of linearised and leading order weakly nonlinear Navier Stokes

In this section I derive the equations for small linearised and weakly nonlinear perturbations in a cylindrical duct without any assumptions on the thickness of the boundary layer δ or any asymptotic matching. To do this I expand the governing equations (2.1) to $\mathcal{O}(\varepsilon^2)$ ignoring terms that are $\mathcal{O}(1)$ as they are expected to be satisfied by the base flow.

We write all acoustic terms as $\tilde{u} = \varepsilon \tilde{u}_1 + \varepsilon^2 \tilde{u}_2$. Using the nonlinear multiplication rule from §3.5 we will have a 'zero' mode and a 'double' mode for $\tilde{u}_2$ which will be considered separately.

We start by considering the equation of state (2.1e), note $p = 1/\gamma$ in base flow,

$$T = \frac{\gamma p}{(\gamma - 1)\rho}.$$

Considering the leading order linear terms and linear order quadratic terms, this becomes

$$\tilde{\rho}_1 + \varepsilon \tilde{\rho}_2 = \frac{\tilde{p}_1 \gamma}{(\gamma - 1)T} - \frac{\tilde{T}_1}{(\gamma - 1)T^2} + \frac{\varepsilon \tilde{p}_2 \gamma}{(\gamma - 1)T} - \frac{\varepsilon \tilde{T}_2}{(\gamma - 1)T^2} - \frac{\varepsilon \tilde{p}_1 \tilde{T}_1 \gamma}{(\gamma - 1)T^2} + \frac{\varepsilon \tilde{T}_1 \tilde{T}_1}{(\gamma - 1)T^3}.$$

Matching orders of ε gives at leading order gives

$$\tilde{\rho}_1 = \frac{\tilde{p}_1 \gamma}{(\gamma - 1)T} - \frac{\tilde{T}_1}{(\gamma - 1)T^2}, \tag{B.7}$$

while at order ε^2, taking only 'zero' mode gives

$$\tilde{\rho}_2 = \frac{\tilde{p}_2 \gamma}{(\gamma - 1)T} - \frac{\tilde{T}_2}{(\gamma - 1)T^2} - \frac{\tilde{p}_1^\star \tilde{T}_1 \gamma}{4(\gamma - 1)T^2} + \frac{\tilde{T}_1^\star \tilde{T}_1}{4(\gamma - 1)T^3}. \tag{B.8}$$

In the double mode case the result is the same but without the complex conjugates

$$\tilde{\rho}_2 = \frac{\tilde{p}_2 \gamma}{(\gamma - 1)T} - \frac{\tilde{T}_2}{(\gamma - 1)T^2} - \frac{\tilde{p}_1 \tilde{T}_1 \gamma}{4(\gamma - 1)T^2} + \frac{\tilde{T}_1 \tilde{T}_1}{4(\gamma - 1)T^3}. \tag{B.9}$$

Now we let $K = k - k^\star$, $\Omega = \omega - \omega^\star$ and $\mathcal{M} = 0$ for the zero mode and $K = 2k$, $\Omega = 2\omega$ and $\mathcal{M} = 2m$ for the double mode as in §3.5. Now we consider the equation of mass conservation (2.1a)

$$\rho_t + (\rho u)_x + \frac{1}{r}(r\rho v)_r + \frac{1}{r}(\rho w)_\theta = 0.$$

At leading order, $\mathcal{O}(\varepsilon)$, this becomes

$$i(\omega - Uk)\tilde{\rho}_1 - ik\rho\tilde{u}_1 + \frac{1}{r}\rho\tilde{v}_1 + \rho_r\tilde{v}_1 + \rho\tilde{v}_{1r} - \frac{im}{r}\rho\tilde{w}_1 = 0,$$

while at $\mathcal{O}(\varepsilon^2)$ we get

$$i(\Omega - UK)\tilde{\rho}_2 - iK\rho\tilde{u}_2 + \frac{(r\rho\tilde{v}_2)_r}{r} - \frac{i\mathcal{M}}{r}\rho\tilde{w}_2 = \frac{iK}{4}\tilde{\rho}_1^\star \tilde{u}_1 - \frac{(r(\tilde{\rho}_1^\star \tilde{v}_1))_r}{4r} + \frac{i\mathcal{M}}{4r}\tilde{\rho}_1^\star \tilde{w}_1,$$

with the complex conjugate '$\star$' ignored for the double mode. Substituting for ρ and $\tilde{\rho}$ gives

$$i(\omega - Uk)\gamma\tilde{p}_1 - \frac{i(\omega - Uk)}{T}\tilde{T}_1 - ik\tilde{u}_1 + \frac{1}{r}\tilde{v}_1 - \frac{T_r}{T}\tilde{v}_1 + \tilde{v}_{1r} - \frac{im}{r}\tilde{w}_1 = 0, \tag{B.10}$$

and

$$
\begin{aligned}
i(\Omega - UK) & \left(\gamma\tilde{p}_2 - \frac{\tilde{T}_2}{T}\right) - iK\tilde{u}_2 + \frac{1}{r}\tilde{v}_2 - \frac{T_r}{T}\tilde{v}_2 + \tilde{v}_{2r} - \frac{i\mathcal{M}}{r}\tilde{w}_2 \\
& = i(\Omega - UK)\left(\frac{\gamma}{4T}\tilde{p}_1^\star\tilde{T}_1 - \frac{\tilde{T}_1^\star\tilde{T}_1}{4T^2}\right) + \frac{iK}{4}\tilde{u}_1\left(\gamma\tilde{p}_1^\star - \frac{\tilde{T}_1^\star}{T}\right) + \frac{i\mathcal{M}}{4r}\tilde{w}_1\left(\gamma\tilde{p}_1^\star - \frac{\tilde{T}_1^\star}{T}\right) \quad \text{(B.11)} \\
& - \left(\frac{\tilde{v}_1}{4r} + \frac{\tilde{v}_{1r}}{4}\right)\left(\gamma\tilde{p}_1^\star - \frac{\tilde{T}_1^\star}{T}\right) - \frac{\tilde{v}_1}{4}\left(\gamma\tilde{p}_{1r}^\star - \frac{\gamma\tilde{p}_1^\star T_r}{T} - \frac{\tilde{T}_{1r}^\star}{T} + \frac{2\tilde{T}_1^\star T_r}{T^2}\right),
\end{aligned}
$$

where complex conjugates only apply to the zero mode case.

Now the mean flow viscosity μ is given by equation (2.2)

$$\mu = \frac{T}{T_0 Re} = \xi\delta^2(\gamma - 1)T, \tag{B.12}$$

and similarly the viscosity perturbation is given by

$$\tilde{\mu} = \xi\delta^2(\gamma - 1)\tilde{T}. \tag{B.13}$$

Using this we can then write down the linear viscous stress tensor perturbation $\tilde{\sigma}_{1ij}$

$$
\begin{aligned}
\tilde{\sigma}_{1ij} = &\begin{pmatrix} -2ik\mu\tilde{u}_1 & \mu(\tilde{u}_{1r} - ik\tilde{v}_1) + U_r\tilde{\mu}_1 & -\mu\left(\frac{im}{r}\tilde{u}_1 + ik\tilde{w}_1\right) \\ \cdot & 2\mu\tilde{v}_{1r} & \mu\left(-\frac{im}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r}\right) \\ \cdot & \cdot & \frac{2\mu}{r}\left(-im\tilde{w}_1 + \tilde{v}_1\right) \end{pmatrix} \\
& + \mu\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1\right]\delta_{ij},
\end{aligned}
$$

where the '$\cdot$' terms have not been written down as the stress tensor is symmetric and so it is fully determined by its upper triangular part. Similarly the nonlinear viscous stress tensor

$\tilde{\sigma_{2ij}}$ is given by

$$\sigma_{2ij} - \tilde{\sigma}_{ij} = \frac{\tilde{\mu}_1^\star}{4} \begin{pmatrix} -2ik\tilde{u}_1 & (\tilde{u}_{1r} - ik\tilde{v}_1) & -(\frac{im}{r}\tilde{u}_1 + ik\tilde{w}_1) \\ \cdot & 2\tilde{v}_{1r} & (-\frac{im}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r}) \\ \cdot & \cdot & \frac{2}{r}(-im\tilde{w}_1 + \tilde{v}_1) \end{pmatrix}$$
$$+ \frac{\tilde{\mu}_1^\star}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1\right]\delta_{ij},$$

where $\tilde{\sigma}_{ij}$ correspondsto σ_{1ij} with all linear '1' perturbation terms replaced by the nonlinear '2' quantities with $k \to K$, $m \to \mathcal{M}$ and $\omega \to \Omega$. The complex conjugates are only applied in the zero mode case.

Now that we have the viscous stress tensor for both the linear and nonlinear acoustics we can consider streamwise momentum equation

$$\rho\left(u_t + uu_x + vu_r + \frac{1}{r}wu_\theta\right) = -p_x + (\sigma_{11})_x + \frac{1}{r}(r\sigma_{12})_r + \frac{1}{r}(\sigma_{13})_\theta.$$

Linearising this equation we obtain

$$\frac{i(\omega - Uk)}{(\gamma-1)T}\tilde{u}_1 + \frac{U_r}{(\gamma-1)T}\tilde{v}_1 - ik\tilde{p}_1 + ik\mu\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1\right]$$
$$- \mu_r(\tilde{u}_{1r} - ik\tilde{v}_1) - \mu\left(\tilde{u}_{1rr} - ik\tilde{v}_{1r} + \frac{\tilde{u}_{1r}}{r} - \frac{ik\tilde{v}_1}{r}\right) \quad \text{(B.14)}$$
$$+ 2k^2\mu\tilde{u}_1 + \frac{\mu km}{r}\tilde{w}_1 + \frac{\mu m^2}{r^2}\tilde{u}_1 - U_r\left(\tilde{\mu}_{1r} + \frac{\tilde{\mu}_1}{r}\right) - U_{rr}\tilde{\mu}_1 = 0,$$

and for the nonlinear terms we have the same equation now forced by terms that are quadratic in the linear acoustic quantities

$$L_2^x = \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\tilde{p}_1^\star\gamma}{4(\gamma-1)T}\right)(i(\omega - Uk)\tilde{u}_1 + U_r\tilde{v}_1) + \left(\frac{\tilde{\mu}_1^\star}{4r} + \frac{\tilde{\mu}_{1r}^\star}{4}\right)(\tilde{u}_{1r} - ik\tilde{v}_1)$$
$$- \frac{Kk\tilde{u}_1\tilde{\mu}_1^\star}{2} - \frac{iK\tilde{\mu}_1^\star}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1\right] - \frac{\mathcal{M}\tilde{\mu}_1^\star}{4r}\left(\frac{m\tilde{u}_1}{r} + k\tilde{w}_1\right)$$
$$+ \frac{\tilde{\mu}_1^\star}{4}(\tilde{u}_{1rr} - ik\tilde{v}_{1r}) + \frac{1}{4(\gamma-1)T}(ik\tilde{u}_1^\star\tilde{u}_1 - \tilde{v}_1^\star\tilde{u}_{1r} + \frac{im}{r}\tilde{w}_1^\star\tilde{u}_1),$$

where L_2^x has the same form as the left hand side of the leading order equation but with 2 quantities and $k \to K$, $m \to \mathcal{M}$ and $\omega \to \Omega$.

Now we consider the radial momentum equation

$$\rho(v_t + uv_x + vv_r + \frac{1}{r}wv_\theta) = \frac{1}{r}\rho w^2 - p_r - \frac{1}{r}\sigma_{33} + (\sigma_{12})_x + \frac{1}{r}(r\sigma_{22})_r + \frac{1}{r}(\sigma_{23})_\theta.$$

For the linear acoustics this becomes

$$\rho i(\omega - Uk)\tilde{v}_1 = -\tilde{p}_{1r} - ik(\mu(\tilde{u}_{1r} - ik\tilde{v}_1) + U_r\tilde{\mu}_1) - \frac{im\mu}{r}(-\frac{im}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r})$$

$$- \frac{2\mu}{r^2}(\tilde{v}_1 - im\tilde{w}_1) - \frac{\mu}{r}\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1\right] + \frac{2\mu}{r}\tilde{v}_{1r} + 2(\mu\tilde{v}_{1r})_r$$

$$+ \mu\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)\left[-\frac{ik}{r}\tilde{u}_1 - ik\tilde{u}_{1r} + \frac{2}{r}\tilde{v}_{1r} + \tilde{v}_{1rr} - \frac{im}{r}\tilde{w}_{1r} + \mu_r(-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1)\right],$$

which simplifies slightly to give

$$\frac{i(\omega - Uk)}{(\gamma - 1)T}\tilde{v}_1 + \tilde{p}_{1r} + \mu(ik\tilde{u}_{1r} + k^2\tilde{v}_1) + ikU_r\tilde{\mu}_1 + \frac{\mu m^2}{r^2}\tilde{v}_1$$

$$+ \frac{im\mu}{r}(\tilde{w}_{1r} - \frac{3\tilde{w}_1}{r}) + \frac{2\mu}{r^2}\tilde{v}_1 - \frac{2\mu}{r}\tilde{v}_{1r} - 2\mu_r\tilde{v}_{1r} - 2\mu\tilde{v}_{1rr}$$

$$- \mu\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_{1r} + \left(\frac{\tilde{v}_1}{r}\right)_r + \tilde{v}_{1rr} - im\left(\frac{\tilde{w}_1}{r}\right)_r - \frac{\mu_r}{\mu}(ik\tilde{u}_1 - \tilde{v}_{1r} - \frac{\tilde{v}_1}{r} + \frac{im}{r}\tilde{w}_1)\right] = 0,$$

$$(B.15)$$

and the forcing for the nonlinear acoustics is

$$L_2^r = \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\tilde{p}_1^\star\gamma}{4(\gamma-1)T}\right)(i(\omega - Uk)\tilde{v}_1) + \frac{(ik\tilde{u}_1^\star\tilde{v}_1 - \tilde{v}_1^\star\tilde{v}_{1r} + \frac{im}{r}\tilde{w}_1^\star\tilde{v}_1)}{4(\gamma-1)T}$$

$$+ \frac{1}{4r(\gamma-1)T}\tilde{w}_1\tilde{w}_1^\star - \frac{iK\tilde{\mu}_1^\star}{4}(\tilde{u}_{1r} - ik\tilde{v}_1)$$

$$- \frac{\tilde{\mu}_1^\star}{4r}\left(\frac{2}{r}(\tilde{v}_1 - im\tilde{w}_1) + \left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1]\right)$$

$$+ \frac{\tilde{\mu}_1^\star\tilde{v}_{1r}}{2r} + \frac{\tilde{\mu}_{1r}^\star\tilde{v}_{1r}}{2} + \frac{\tilde{\mu}_1^\star\tilde{v}_{1rr}}{2} + \frac{\tilde{\mu}_1^\star}{4}\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)\left[-ik\tilde{u}_{1r} - \frac{ik}{r}\tilde{u}_1 + 2\frac{1}{r}\tilde{v}_{1r} + \tilde{v}_{1rr} - \frac{im}{r}\tilde{w}_{1r}\right]$$

$$+ \frac{\tilde{\mu}_{1r}^\star}{4}\left(\frac{\mu_0^{B*}}{\mu_0^*} - \frac{2}{3}\right)[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1] - \frac{i\mathcal{M}\tilde{\mu}_1^\star}{4r}\left(-\frac{im}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r}\right),$$

where L_2^r again has the same form as the linear equation (B.15). The forcing terms here then simplify to

$$
\begin{aligned}
L_2^r = {} & \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\tilde{p}_1^\star \gamma}{4(\gamma-1)T} \right) (\mathrm{i}(\omega-Uk)\tilde{v}_1) + \frac{(\mathrm{i}k\tilde{u}_1^\star \tilde{v}_1 - \tilde{v}_1^\star \tilde{v}_{1r} + \frac{im}{r}\tilde{w}_1^\star \tilde{v}_1)}{4(\gamma-1)T} \\
& + \frac{1}{4r(\gamma-1)T}\tilde{w}_1\tilde{w}_1^\star - \frac{\tilde{\mu}_1^\star}{2r^2}(\tilde{v}_1 - \mathrm{i}m\tilde{w}_1) - \frac{\mathrm{i}K\tilde{\mu}_1^\star}{4}(\tilde{u}_{1r} - \mathrm{i}k\tilde{v}_1) \\
& - \frac{\mathrm{i}\mathcal{M}\tilde{\mu}_1^\star}{4r}\left(-\frac{\mathrm{i}m}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r} \right) + \frac{\tilde{\mu}_1^\star \tilde{v}_{1r}}{2r} + \frac{\tilde{\mu}_{1r}^\star \tilde{v}_{1r}}{2} + \frac{\tilde{\mu}_1^\star \tilde{v}_{1rr}}{2} \\
& + \frac{\tilde{\mu}_{1r}^\star}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[-\mathrm{i}k\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 \right] \\
& + \frac{\tilde{\mu}_1^\star}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[-\mathrm{i}k\tilde{u}_{1r} + \frac{1}{r}\tilde{v}_{1r} + \tilde{v}_{1rr} - \frac{\tilde{v}_1}{r^2} + \frac{\mathrm{i}m}{r^2}\tilde{w} - \frac{\mathrm{i}m}{r}\tilde{w}_{1r} \right].
\end{aligned}
\tag{B.16}
$$

Now we consider the θ-momentum equation

$$
\rho\left(w_t + uw_x + vw_r + \frac{1}{r}ww_\theta \right) = -\frac{1}{r}\rho vw - \frac{1}{r}p_\theta + (\sigma_{13})_x + \frac{1}{r^2}(r^2\sigma_{23})_r + \frac{1}{r}(\sigma_{33})_\theta.
$$

Linearising this equation we obtain the following equation for the linear acoustics

$$
\begin{aligned}
\rho\mathrm{i}(\omega-Uk)\tilde{w}_1 = {} & \frac{\mathrm{i}m}{r}\tilde{p}_1 + \mathrm{i}k\mu\left(\frac{\mathrm{i}m}{r}\tilde{u}_1 + \mathrm{i}k\tilde{w}_1 \right) - \frac{2\mu\mathrm{i}m}{r^2}(\tilde{v}_1 - \mathrm{i}m\tilde{w}_1) \\
& - \frac{\mathrm{i}m\mu}{r}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[-\mathrm{i}k\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{\mathrm{i}m}{r}\tilde{w}_1 \right] \\
& + \mu_r\left(\frac{-\mathrm{i}m}{r}\tilde{v}_1 + \tilde{w}_{1r} - \frac{\tilde{w}_1}{r} \right) + \mu\left(\frac{-\mathrm{i}m}{r}\tilde{v}_{1r} - \frac{\mathrm{i}m}{r^2}\tilde{v}_1 + \tilde{w}_{1rr} + \frac{\tilde{w}_{1r}}{r} - \frac{\tilde{w}_1}{r^2} \right),
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
& \frac{\mathrm{i}(\omega-Uk)}{(\gamma-1)T}\tilde{w}_1 - \frac{\mathrm{i}m}{r}\tilde{p}_1 + \mu\left(\frac{km}{r}\tilde{u}_1 + k^2\tilde{w}_1 \right) + \frac{3\mu\mathrm{i}m}{r^2}\tilde{v}_1 + \frac{2\mu m^2}{r^2}\tilde{w}_1 \\
& + \mu\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[\frac{km}{r}\tilde{u}_1 + \frac{\mathrm{i}m}{r}\tilde{v}_{1r} + \frac{\mathrm{i}m\tilde{v}_1}{r^2} + \frac{m^2}{r^2}\tilde{w}_1 \right] \\
& + \mu_r\left(\frac{\mathrm{i}m}{r}\tilde{v}_1 - \tilde{w}_{1r} + \frac{\tilde{w}_1}{r} \right) + \mu\left(\frac{\mathrm{i}m}{r}\tilde{v}_{1r} - \tilde{w}_{1rr} - \frac{\tilde{w}_{1r}}{r} + \frac{\tilde{w}_1}{r^2} \right) = 0,
\end{aligned}
\tag{B.17}
$$

and for the nonlinear acoustics we have L_2^θ has the same form as (B.17) and the forcing is given by

$$
\begin{aligned}
L_2^\theta = {}& \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\gamma \tilde{p}_1^\star}{4(\gamma-1)T} \right) i(\omega - Uk)\tilde{w}_1 + iK \frac{\tilde{\mu}_1^\star}{4} \left(\frac{im}{r}\tilde{u}_1 + ik\tilde{w}_1 \right) \\
& + \frac{1}{4(\gamma-1)T} \left(ik\tilde{u}_1^\star \tilde{w}_1 - \tilde{v}_1^\star \tilde{w}_{1r} + \frac{im}{r}\tilde{w}_1^\star \tilde{w}_1 - \frac{1}{r}\tilde{v}_1^\star \tilde{w}_1 \right) \\
& - \frac{\tilde{\mu}_1^\star}{4} \left(\frac{im}{r}\tilde{v}_{1r} + \frac{im}{r^2}\tilde{v}_1 - \tilde{w}_{rr} + \frac{\tilde{w}_1}{r^2} - \frac{\tilde{w}_{1r}}{r} \right) - \frac{\tilde{\mu}_{1r}^\star}{4} \left(\frac{im\tilde{v}_1}{r} - \tilde{w}_{1r} + \frac{\tilde{w}_1}{r} \right) \\
& - \frac{i\mathcal{M}\tilde{\mu}_1^\star}{2r^2}(\tilde{v}_1 - im\tilde{w}_1) - \frac{i\mathcal{M}\tilde{\mu}_1^\star}{4r} \left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right) \left[-ik\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{im}{r}\tilde{w}_1 \right].
\end{aligned}
\tag{B.18}
$$

Finally we consider the energy equation (2.1d), noting that $\kappa = \mu/Pr$ so we can use the results (B.12) and (B.13) to substitute for κ. The equation is

$$
\begin{aligned}
&\rho\left(T_t + uT_x + vT_r + \frac{1}{r}wT_\theta\right) = \left(p_t + up_x + vp_r + \frac{1}{r}wp_\theta\right) + (\kappa T_x)_x + \frac{1}{r}(r\kappa T_r)_r + \frac{1}{r^2}(\kappa T_\theta)_\theta \\
&+ \sigma_{11}u_x + \sigma_{12}(u_r + v_x) + \sigma_{13}\left(\frac{1}{r}u_\theta + w_x\right) + \sigma_{22}v_r + \sigma_{23}\left(\frac{1}{r}v_\theta + r\left(\frac{w}{r}\right)_r\right) + \frac{1}{r}\sigma_{33}(w_\theta + v).
\end{aligned}
$$

When we linearise this equation we get

$$
\begin{aligned}
\rho i(\omega - Uk)\tilde{T}_1 + \rho\tilde{v}_1 T_r = {}& i(\omega - Uk)\tilde{p}_1 - k^2\kappa\tilde{T}_1 + \frac{1}{r}(r\kappa\tilde{T}_{1r} + r\tilde{\kappa}_1 T_r)_r \\
& - \frac{m^2}{r^2}\kappa\tilde{T}_1 + U_r^2\tilde{\mu}_1 + 2\mu U_r(\tilde{u}_{1r} - ik\tilde{v}_1),
\end{aligned}
$$

which can then be simplified to give

$$
\begin{aligned}
& \frac{i(\omega - Uk)}{(\gamma-1)T}\tilde{T}_1 + \frac{\tilde{v}_1 T_r}{(\gamma-1)T} - i(\omega - Uk)\tilde{p}_1 + k^2\kappa\tilde{T}_1 - \frac{1}{r}(r\kappa\tilde{T}_{1r} + r\tilde{\kappa}_1 T_r)_r \\
& + \frac{m^2}{r^2}\kappa\tilde{T}_1 - U_r^2\tilde{\mu}_1 - 2\mu U_r(\tilde{u}_{1r} - ik\tilde{v}_1) = 0.
\end{aligned}
\tag{B.19}
$$

For the nonlinear acoustics we have the linear operator of the nonlinear terms L_2^T forced by

$$
\begin{aligned}
L_2^T = {} & \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\gamma \tilde{p}_1^\star}{4(\gamma-1)T} \right) (\mathrm{i}(\omega - Uk)\tilde{T}_1 + \tilde{v}_1 T_r) + \frac{1}{4r}(r\tilde{\kappa}_1^\star \tilde{T}_{1r})_r - \frac{\mathcal{M}m}{4r^2}\tilde{\kappa}_1^\star \tilde{T}_1 \\
& - \frac{\tilde{v}_1^\star}{4}\left(\frac{\tilde{T}_{1r}}{(\gamma-1)T} - \tilde{p}_{1r} \right) - \frac{Kk}{4}\tilde{\kappa}_1^\star \tilde{T}_1 + \frac{1}{4}(\mathrm{i}k\tilde{u}_1^\star + \frac{\mathrm{i}m}{r}\tilde{w}_1^\star)\left(\frac{\tilde{T}_1}{(\gamma-1)T} - \tilde{p}_1 \right) \\
& + \frac{1}{4}(\tilde{u}_{1r}^\star - (\mathrm{i}k)^\star \tilde{v}_1^\star)(\mu(\tilde{u}_{1r} - \mathrm{i}k\tilde{v}_1) + U_r \tilde{\mu}_1) + \frac{1}{4}U_r \tilde{\mu}_1^\star(\tilde{u}_{1r} - \mathrm{i}k\tilde{v}_1) \\
& - \frac{\mu}{4}\left(-\frac{(\mathrm{i}m)^\star}{r}\tilde{u}_1^\star - (\mathrm{i}k)^\star \tilde{w}_1^\star \right)\left(\frac{\mathrm{i}m}{r}\tilde{u}_1 + \mathrm{i}k\tilde{w}_1 \right) \\
& - \frac{\mu}{4}\left(-\frac{(\mathrm{i}m)^\star}{r}\tilde{v}_1^\star + \tilde{w}_{1r}^\star - \frac{\tilde{w}_1^\star}{r} \right)\left(\frac{\mathrm{i}m}{r}\tilde{v}_1 - \tilde{w}_{1r} + \frac{\tilde{w}_1}{r} \right) \\
& + \frac{\mathrm{i}k(\mathrm{i}k)^\star \mu}{2}\tilde{u}_1^\star \tilde{u}_1 + \frac{\mu}{2}\tilde{v}_{1r}^\star \tilde{v}_{1r} + \frac{\mu}{2r^2}(\tilde{v}_1^\star - (\mathrm{i}m)^\star \tilde{w}_1^\star)(\tilde{v}_1 - \mathrm{i}mw_1) \\
& + \frac{\mu}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[-\mathrm{i}k\tilde{u}_1 + \tilde{v}_{1r} + \frac{\tilde{v}_1}{r} - \frac{\mathrm{i}m\tilde{w}_1}{r} \right]\left(-(\mathrm{i}k)^\star \tilde{u}_1^\star + \tilde{v}_{1r}^\star + \frac{\tilde{v}_1^\star}{r} - \frac{(\mathrm{i}m)^\star \tilde{w}_1^\star}{r} \right),
\end{aligned}
$$

which can be expanded and written as

$$
\begin{aligned}
L_2^T = {} & \left(\frac{\tilde{T}_1^\star}{4(\gamma-1)T^2} - \frac{\gamma \tilde{p}_1^\star}{4(\gamma-1)T} \right) (\mathrm{i}(\omega - Uk)\tilde{T}_1 + \tilde{v}_1 T_r) \\
& + \frac{1}{4}(\mathrm{i}k\tilde{u}_1^\star + \frac{\mathrm{i}m}{r}\tilde{w}_1^\star)\left(\frac{\tilde{T}_1}{(\gamma-1)T} - \tilde{p}_1 \right) \\
& - \frac{\tilde{v}_1^\star}{4}\left(\frac{\tilde{T}_{1r}}{(\gamma-1)T} - \tilde{p}_{1r} \right) - \frac{Kk}{4}\tilde{\kappa}_1^\star \tilde{T}_1 - \frac{\mathcal{M}m}{4r^2}\tilde{\kappa}_1^\star \tilde{T}_1 + \frac{1}{4r}(r\tilde{\kappa}_1^\star \tilde{T}_{1r})_r + \frac{1}{2}(\tilde{u}_{1r} - \mathrm{i}k\tilde{v}_1)U_r \tilde{\mu}_1^\star \\
& + \frac{\mu}{4}(\tilde{u}_{1r}^\star \tilde{u}_{1r} + \mathrm{i}k(\mathrm{i}k)^\star \tilde{v}_1^\star \tilde{v}_1 - 2(\mathrm{i}k)^\star \tilde{v}_1^\star \tilde{u}_{1r}) + \frac{\mu \mathrm{i}(\mathrm{i})^\star}{4}\left(\frac{m^2}{r^2}\tilde{u}_1^\star \tilde{u}_1 + kk^\star \tilde{w}_1^\star \tilde{w}_1 + \frac{2mk}{r}\tilde{u}_1^\star \tilde{w}_1 \right) \\
& + \frac{\mu}{4}\left(\frac{(\mathrm{i}m)(\mathrm{i}m)^\star}{r^2}\tilde{v}_1^\star \tilde{v}_1 + \tilde{w}_{1r}^\star \tilde{w}_{1r} + \frac{\tilde{w}_1^\star \tilde{w}_1}{r^2} - \frac{2(\mathrm{i}m)^\star}{r}\tilde{v}_1^\star \tilde{w}_{1r} + \frac{2\mathrm{i}m}{r^2}\tilde{w}_1^\star \tilde{v}_1 - \frac{2}{r}\tilde{w}_{1r}^\star \tilde{w}_1 \right) \\
& + \frac{\mathrm{i}k(\mathrm{i}k)^\star \mu}{2}\tilde{u}_1^\star \tilde{u}_1 + \frac{\mu}{2}\tilde{v}_{1r}^\star \tilde{v}_{1r} + \frac{\mu}{2r^2}(\tilde{v}_1^\star \tilde{v}_1 + \mathrm{i}m(\mathrm{i}m)^\star \tilde{w}_1^\star \tilde{w}_1 - 2\mathrm{i}m\tilde{w}_1 \tilde{v}_1^\star) \\
& + \frac{\mu}{4}\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3} \right)\left[\mathrm{i}k(\mathrm{i}k)^\star \tilde{u}_1 \tilde{u}_1^\star + \tilde{v}_{1r}\tilde{v}_{1r}^\star + \frac{\tilde{v}_1 \tilde{v}_1^\star}{r^2} + \frac{\mathrm{i}m(\mathrm{i}m)^\star \tilde{w}_1^\star \tilde{w}_1}{r^2} - 2(\mathrm{i}k)^\star \tilde{u}_1^\star \tilde{v}_{1r} - \frac{2\mathrm{i}k\tilde{u}_1 \tilde{v}_1^\star}{r} \right. \\
& \left. + \frac{2(\mathrm{i}m)^\star \mathrm{i}k\tilde{u}_1 \tilde{w}_1^\star}{r} + \frac{2\tilde{v}_{1r}\tilde{v}_1^\star}{r} - \frac{2(\mathrm{i}m)^\star \tilde{v}_{1r}\tilde{w}_1^\star}{r} - \frac{2\mathrm{i}m\tilde{v}_1^\star \tilde{w}_1}{r^2} \right].
\end{aligned}
$$

B.3 Boundary conditions for linearised and weakly nonlinear Navier Stokes

In this section I will derive the boundary conditions for the linearised and weakly nonlinear Navier Stokes. Both the linearised and weakly nonlinear Navier Stokes are second order in $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{T}$ this means that we must supply two boundary conditions for each of these terms, while we only need one boundary condition for $\tilde{p}$. At the wall the no-slip condition (2.3) gives $\tilde{u}(1) = \tilde{w}(1) = 0$ and the acoustic thermal boundary condition (2.6) at the wall gives $\tilde{T}(1) = 0$. At the wall we also set $\tilde{v}_1(1) = \tilde{v}_0$ where $\tilde{v}_0$ is an arbitrary value that sets the amplitude and phase of our perturbation. We can then check whether the impedance condition is satisfied for the given ω, k and m. For the nonlinear normal velocity we have $\tilde{v}_2(1) = 0$ for the zero mode, as there can be no net penetration through the wall, and for the double mode we have $Z(\Omega)\tilde{v}_2(1) = \tilde{p}_2(1)$.

To get another boundary condition for each term we must require that our solutions are regular at the origin. To do this we assume each acoustic quantity has a power series solution near the origin and we require that all singular terms of order $\mathscr{O}\left(\frac{1}{r}\right)$ or $\mathscr{O}\left(\frac{1}{r^2}\right)$ in the equations must cancel.

The boundary conditions we derive are different depending on the value of m, we first consider the similar cases $m = 0$ for the linear acoustics and $\mathcal{M} = 0$ for the nonlinear acoustics.

B.3.1 Linear $m = 0$ and nonlinear $\mathcal{M} = 0$

Keeping only leading order, $\mathscr{O}\left(\frac{1}{r}\right)$, terms in r, equation (B.10) gives

$$\frac{\tilde{v}_1(0)}{r} = 0,$$

so

$$\tilde{v}_1 = 0 \quad \text{at} \quad r = 0, \tag{B.20}$$

and for the nonlinear acoustics we have

$$\frac{\tilde{v}_2(0)}{r} = -\frac{\tilde{v}_1(0)}{4r}\left(\gamma \tilde{p}_1^\star(0) - \frac{\tilde{T}_1^\star(0)}{T}\right).$$

Using the linear boundary condition (B.20) this is zero for $m = 0$ but for $m \neq 0$ this may not be true so the nonlinear boundary condition is

$$\tilde{v}_2 = -\frac{\tilde{v}_1}{4}\left(\gamma \tilde{p}_1^\star - \tilde{T}_1^\star(\gamma - 1)\right) \quad \text{at} \quad r = 0. \tag{B.21}$$

Now the leading order terms of equation (B.14) give

$$\mu \frac{\tilde{u}_{1r}(0)}{r} - U_r \frac{\tilde{u}_1(0)}{r} = 0,$$

but $\lim_{r \to 0} U_r = 0$ as U only varies radially within the boundary layer, so we have

$$\tilde{u}_{1r} = 0 \quad \text{at} \quad r = 0. \tag{B.22}$$

Similarly for the nonlinear streamwise acoustic velocity we have

$$\tilde{u}_{2r} = 0 \quad \text{at} \quad r = 0. \tag{B.23}$$

For the radial momentum equation (B.15), we find that both the order $\mathcal{O}(\frac{1}{r^2})$ and order $\mathcal{O}(\frac{1}{r})$ terms cancel since $\tilde{v}_1(0) = 0$, so we must consider the balance of the order $\mathcal{O}(1)$ terms

$$\tilde{p}_{1r}(0) + 2\mu\left(\frac{\tilde{v}_1(0)}{r^2} - \frac{\tilde{v}_{1r}(0)}{r} - \tilde{v}_{1rr}(0)\right) - \mu\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right)\left(\frac{1}{r}\tilde{v}_{1r}(0) - \frac{\tilde{v}_1(0)}{r^2} + \tilde{v}_{1rr}(0)\right) = 0,$$

but $\tilde{v}_1 = a_1 r + a_2 r^2 + \dots$ near $r = 0$, so this becomes

$$\tilde{p}_{1r} = \frac{3}{2}\mu\tilde{v}_{1rr}\left(\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right) + 2\right) \quad \text{at} \quad r = 0. \tag{B.24}$$

Similarly for the nonlinear acoustics we have

$$\tilde{p}_{2r} - \frac{3}{2}\mu\tilde{v}_{2rr}\left(\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right) + 2\right) = \frac{3}{8}\tilde{\mu}_1^\star\tilde{v}_{1rr}\left(\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right) + 2\right) \quad \text{at} \quad r = 0, \tag{B.25}$$

if $m = 0$ and

$$\tilde{p}_{2r} - \frac{3}{2}\mu\tilde{v}_{2rr}\left(\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right) + 2\right) = -\frac{\gamma\mathrm{i}(\omega - Uk)\tilde{p}_1^\star\tilde{v}_1}{4} + \frac{3\mathrm{i}m}{8}\tilde{w}_1^\star\tilde{v}_{1r}$$
$$+ \frac{\tilde{\mu}_{1r}^\star\tilde{v}_{1r}}{8}\left(\left(\frac{\mu_0^{B^\star}}{\mu_0^\star} - \frac{2}{3}\right)(4 - m^2) + (4 + 2m^2)\right) \quad \text{at} \quad r = 0, \tag{B.26}$$

if $m \neq 0$. The leading order terms of the azimuthal momentum equation (B.17) are

$$\frac{1}{r^2}\tilde{w}_1(0) - \frac{1}{r}\tilde{w}_{1r}(0) = 0,$$

so $\tilde{w}_1 = a_1 r + ...$, which gives

$$\tilde{w}_1 = 0 \quad \text{at} \quad r = 0, \tag{B.27}$$

and similarly for the nonlinear term we have

$$\tilde{w}_2 = 0 \quad \text{at} \quad r = 0. \tag{B.28}$$

Lastly the energy equation (B.19) gives

$$\frac{1}{r}(\kappa\tilde{T}_{1r}(0) + \tilde{\kappa}_1(0)T_r) = 0,$$

but $T_r \to 0$ near $r = 0$ since it only varies radially within the boundary layer, so we have

$$\tilde{T}_{1r} = 0 \quad \text{at} \quad r = 0. \tag{B.29}$$

Similarly for the nonlinear temperature term we have

$$\tilde{T}_{2r} = 0 \quad \text{at} \quad r = 0. \tag{B.30}$$

B.3.2 $m \neq 0$ **linear**

Now we consider the case of $m \neq 0$ in the same way. Beginning with the linear acoustics. This time equation (B.10) gives

$$\frac{\tilde{v}_1(0)}{r} - \frac{im\tilde{w}_1(0)}{r} = 0,$$

so

$$\tilde{v}_1 = im\tilde{w}_1 \quad \text{at} \quad r = 0. \tag{B.31}$$

The streamwise momentum equation (B.14) at order $\mathcal{O}\left(\frac{1}{r^2}\right)$ gives

$$\tilde{u}_1 = 0 \quad \text{at} \quad r = 0, \tag{B.32}$$

and if we also consider the order $\mathcal{O}(\frac{1}{r})$ terms we have

$$\mu km\tilde{w}_1(0) + \mu m^2\tilde{u}_{1r}(0) + \mu\left(-\tilde{u}_{1r}(0) + ik\tilde{v}_1(0)\right) - U_r\tilde{u}_1(0) = 0,$$

but $\lim_{r\to 0} U_r = 0$. So this gives

$$(1-m^2)\tilde{u}_{1r} = ik\tilde{v}_1 + km\tilde{w}_1 = 0 \quad \text{at} \quad r = 0, \tag{B.33}$$

which gives no information for $m = 1$. Now if we consider the radial momentum equation (B.15) at order $\mathcal{O}\left(\frac{1}{r^2}\right)$ we have

$$-im\mu\tilde{v}_1(0) - \mu\tilde{w}_1(0) = 0,$$

which gives

$$(m^2-1)\tilde{w}_1 = 0 \quad \text{at} \quad r = 0.$$

For $m = 1$ this is true identically and so gives no extra information, but for $m > 1$ this gives

$$\tilde{v}_1 = \tilde{w}_1 = 0 \quad \text{at} \quad r = 0 \quad \text{for } m > 1. \tag{B.34}$$

To find a boundary condition valid for $m = 1$ we continue and consider the terms that are order $\mathcal{O}\left(\frac{1}{r}\right)$. This gives

$$m^2\tilde{v}_{1r}(0) - 2im\tilde{w}_1(0) - \left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)(\tilde{v}_{1r}(0) - \tilde{v}_{1r}(0) + im\tilde{w}_{1r}(0) - im\tilde{w}_{1r}(0)) = 0,$$

which then simplifies to give

$$m\tilde{v}_{1r} = 2i\tilde{w}_{1r} \quad \text{at} \quad r = 0 \quad \text{for } m \geq 1, \tag{B.35}$$

a boundary condition that is valid for $m = 1$ as well as $m > 1$. The $\mathcal{O}\left(\frac{1}{r^2}\right)$ terms of the azimuthal momentum equation (B.17) give

$$\mu(1-m^2)\tilde{w}_1(0) = 0,$$

so $\tilde{w}_1 = 0$ for $m > 1$ as before, and the expression is true identically for $m = 1$. At order $\mathcal{O}\left(\frac{1}{r}\right)$, we have

$$-im\tilde{p}_1(0) + 4\mu im\tilde{v}_{1r}(0) + 2\mu m^2\tilde{w}_{1r}(0) + \mu\left(\frac{\mu_0^{B^*}}{\mu_0^*} - \frac{2}{3}\right)(2im\tilde{v}_{1r}(0) + m^2\tilde{w}_{1r}(0)) = 0,$$

which can then be simplified to give a boundary condition for $\tilde{p}_1$

$$\tilde{p}_1 = \frac{\mu}{2}(4-m^2)\tilde{v}_{1r}\left(2+\left(\frac{\mu_0^{B^*}}{\mu_0^*}-\frac{2}{3}\right)\right) \quad \text{at} \quad r=0 \quad \text{for } m \geq 1. \tag{B.36}$$

Finally the singular terms of the linear energy equation (B.19) give

$$\frac{m^2}{r^2}\kappa\tilde{T}_1 - \frac{1}{r}\kappa\tilde{T}_{1r} = 0,$$

so

$$\tilde{T}_1 = 0 \quad \text{at} \quad r=0 \quad \text{for } m \geq 1. \tag{B.37}$$

We now have a consistent set of boundary conditions for the linear acoustics that are valid for all $m \geq 1$.

B.3.3 $m \neq 0$ nonlinear

To find the boundary conditions for the nonlinear acoustics with $\mathcal{M} \neq 0$ we start with the expansion of equation (B.11), this gives

$$\tilde{v}_2 - i\mathcal{M}\tilde{w}_2 = -\frac{(i\mathcal{M}\tilde{w}_1 - \tilde{v}_1)}{4}\gamma\tilde{p}_1^\star \quad \text{at} \quad r=0, \tag{B.38}$$

which is valid for all m, note that the right hand side is zero for $m=0$. The streamwise momentum equation (B.14) at order $\mathcal{O}\left(\frac{1}{r^2}\right)$ for the nonlinear acoustics gives

$$\mu m^2 \tilde{u}_2 = -\frac{\mathcal{M}\mu_1^\star m\tilde{u}_1}{4} = 0 \quad \text{at} \quad r=0.$$

Similarly to the linear acoustics we find that the leading order terms of both the radial and azimuthal equations give no extra information in the case $\mathcal{M} = 1$ so we must expand to next order. For the radial momentum equation this gives

$$\mathcal{M}\tilde{v}_{2r} - 2i\tilde{w}_{2r} = -\frac{(\gamma-1)\tilde{T}_{1r}^\star}{2\mathcal{M}}(\tilde{v}_{1r} - im\tilde{w}_{1r}) \quad \text{at} \quad r=0, \tag{B.39}$$

while if we consider the azimuthal momentum equation at order $\mathscr{O}(\frac{1}{r})$ we again find the boundary condition for $\tilde{p}$

$$\tilde{p}_2 - \frac{\mu}{2}(4-\mathcal{M}^2)\tilde{v}_{2r}\left(2+\left(\frac{\mu_0^{B^*}}{\mu_0^*}-\frac{2}{3}\right)\right) = -\mathrm{i}m\tilde{v}_1\tilde{\mu}_{1r}^{\star}+\frac{\tilde{w}_1^{\star}\tilde{w}_1}{4}(\mathrm{i}m-(\mathrm{i}m)^{\star}) \quad \text{at} \quad r=0, \tag{B.40}$$

and finally the energy equation gives

$$\tilde{T}_2 = 0 \quad \text{at} \quad r=0. \tag{B.41}$$

Now that we have the boundary conditions for all cases, we can see that, as all terms involving μ are expected to be $\mathscr{O}(\delta^2)$ smaller than the other terms, that these boundary conditions are consistent with the expected Bessel function solutions.

Appendix C

C.1 Airy function integral

The Airy functions are the two independent solutions to the equation [9, Eq. 9.2.1]

$$y''(x) = xy,$$

with $\text{Ai}(x) \to 0$ as $x \to \infty$ and $\text{Bi}(x) \to \infty$ as $x \to \infty$. We can then use this to write

$$\text{Ai}'(xb) = b^2 \int_0^x u\text{Ai}(ub)\,\mathrm{d}u,$$

where we have also used the chain rule here to include the constant b in the equation. This equation can then be used to transform the integral

$$I(y) = \int_0^y \frac{\frac{\text{Ai}'(xb)}{\text{Ai}'(0)} - 1}{x^2}\,\mathrm{d}x.$$

We integrate by parts, integrating the denominator $1/x^2$ and differentiating the numerator. This gives

$$I(y) = -\left[\frac{\frac{\text{Ai}'(xb)}{\text{Ai}'(0)} - 1}{x}\right]_0^y + \frac{b^2}{\text{Ai}'(0)} \int_0^y \text{Ai}(xb)\,\mathrm{d}x,$$

but $\text{Ai}'(x) \sim \text{Ai}'(0) + \mathscr{O}(x^2)$ for small x [9, Eq. 9.4.2]. So this becomes

$$I(y) = -\frac{\frac{\text{Ai}'(yb)}{\text{Ai}'(0)} - 1}{y} + \frac{b^2}{\text{Ai}'(0)} \int_0^y \text{Ai}(xb)\,\mathrm{d}x. \tag{C.1}$$

Now $\tilde{v}_c$ has the form $yI(y)$ and $\tilde{u}_2 \propto \tilde{v}_{cy}$, so we need to calculate $\frac{d}{dy}[yI(y)]$

$$\frac{d}{dy}[yI(y)] = I(y) + yI'(y) = I(y) + \frac{\frac{\mathrm{Ai}'(yb)}{\mathrm{Ai}'(0)} - 1}{y}$$
$$= \frac{b^2}{\mathrm{Ai}'(0)} \int_0^y \mathrm{Ai}(xb)\,\mathrm{d}x.$$

Note that this also holds for the $\mathrm{Bi}(x)$ integral as $\mathrm{Bi}(x)$ also obeys Airy's equation and its series expansion near zero has the same form as that for $\mathrm{Ai}(x)$.

C.2 Small Ω solution

When $\Omega \neq 0$ the equation for $\tilde{v}_2$ is (5.18)

$$\left(\sqrt{\frac{X}{\varepsilon_k}} \frac{\Omega}{K} - yu_\zeta(0) \right) \tilde{v}_{2y} + u_\zeta(0)\tilde{v}_2 - \frac{(\gamma-1)^2 T(0)^2}{\gamma p} \sqrt{\frac{X}{\varepsilon_k}} \frac{1}{iK} \tilde{v}_{2yyy}$$
$$= \frac{-C^a C^b u_\zeta(0)\sqrt{iX\gamma p}}{8\sqrt{\varepsilon_k}\delta_L(\gamma-1)T(0)} \left(iv\sqrt{\omega}e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} + i\omega\sqrt{v}e^{-\frac{y\sqrt{iv\gamma p}}{(\gamma-1)T(0)}} \right) - \frac{iKC^a C^b}{8\delta_L}u_\zeta(0)^2 + E(y).$$

As in chapter §5, this has solution

$$\tilde{v}_2 = -\frac{iKC^a C^b}{8\delta_L}u_\zeta(0)\left(1 - e^{-\frac{y\sqrt{i\omega\gamma p}}{(\gamma-1)T(0)}} - e^{-\frac{y\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}} + \tilde{v}_c \right) + \mathcal{O}(\sqrt{\varepsilon_k}),$$

where $\tilde{v}_c$ is a solution of the homogeneous equation with initial conditions $\tilde{v}_c(0) = Z_1\Omega$ and $\tilde{v}'_c(0) = -\frac{\sqrt{i\omega\gamma p}+\sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)}$, where Z_1 is a known constant. The equation for $\tilde{v}_c$ is now

$$(\kappa - y)\tilde{v}_{cy} + \tilde{v}_c - \frac{\tilde{v}_{cyyy}}{\beta} = 0,$$

where $\kappa = \frac{\Omega}{u_\zeta(0)K}\sqrt{\frac{X}{\varepsilon_k}}$ and β is again given by equation (5.20). Now if we let $z = y - \kappa$ the equation has the same form as equation (5.19), which has the solution (5.24)

$$\tilde{v}_c = a_0 z + a_1 z \int^z \frac{\mathrm{Ai}'(x(-\beta)^{1/3})}{x^2}\,\mathrm{d}x + a_2 z \int^z \frac{\mathrm{Bi}'(x(-\beta)^{1/3})}{x^2}\,\mathrm{d}x.$$

To match to the region II solution we again need $a_2 = 0$. So we have

$$\tilde{v}_c = -a_1 + a_0(y - \kappa) + a_1(y - \kappa) \int_{-\kappa}^{y-\kappa} \frac{\frac{\text{Ai}'(x(-\beta)^{1/3})}{\text{Ai}'(0)} - 1}{x^2} \, dx.$$

If we apply the initial conditions we get the following equations for a_0 and a_1

$$-a_1 - \kappa a_0 = 1 + Z_1 \kappa \sqrt{\varepsilon_k},$$

$$a_0 - \frac{a_1}{\kappa} \left(\frac{\text{Ai}'(-\kappa(-\beta)^{1/3})}{\text{Ai}'(0)} - 1 \right) = -\frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma - 1)T(0)},$$

solving these equations we get the following results

$$a_0 = -\frac{\text{Ai}'(0) \frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)} + \frac{\left(\text{Ai}'(-\kappa(-\beta)^{1/3}) - \text{Ai}'(0)\right)(1 + Z_1\kappa\sqrt{\varepsilon_k})}{\kappa}}{\text{Ai}'(-\kappa(-\beta)^{1/3})},$$

$$a_1 = \frac{\text{Ai}'(0)\kappa \frac{\sqrt{i\omega\gamma p} + \sqrt{-i\omega\gamma p}}{(\gamma-1)T(0)} + \left(\text{Ai}'(-\kappa(-\beta)^{1/3}) - \text{Ai}'(0)\right)(1 + Z_1\kappa\sqrt{\varepsilon_k})}{\text{Ai}'(-\kappa(-\beta)^{1/3})} - 1 - Z_1\kappa\sqrt{\varepsilon_k}.$$

Now $\kappa \sim \frac{\Omega}{\sqrt{\varepsilon_k}}$ and to match to region II we need the asymptotic behaviour of $\tilde{v}_c$ for large y

$$\tilde{v}_c \sim 2a_1 + a_0(y - \kappa) - a_1 \frac{\text{Ai}'((y - \kappa)(-\beta)^{1/3})}{\text{Ai}'(0)} - a_1 \frac{(y - \kappa)}{\kappa} \left(\frac{\text{Ai}'(-\kappa(-\beta)^{1/3})}{\text{Ai}'(0)} - 1 \right)$$

$$- a_1(y - \kappa) \frac{(-\beta)^{2/3}}{\text{Ai}'(0)} \int_0^y \text{Ai}((x - \kappa)(-\beta)^{1/3}) \, dx.$$

For $\kappa = 0$, i.e. $\Omega = 0$, this gives the same asymptotic behaviour as equation (5.28). It is clear that the solution for $\Omega = 0$ given in chapter §5 will be the valid leading order solution provided $\kappa(\beta)^{1/3} \ll 1$. Since $\beta \sim \sqrt{\varepsilon_k}$ this is true for $\kappa \ll \varepsilon_k^{-1/6}$, i.e. $\Omega \ll \varepsilon_k^{1/3}$.

Now for all κ this solution matches to $\sqrt{\varepsilon_k}(y - \kappa)\frac{E}{\delta_L}$ in region II. Figure C.1 shows a plot of $E = \tilde{v}_c/((y - \kappa)\sqrt{\varepsilon_k})$ against κ for different values of ε_k from 10^{-4} up to 10^{-8}. It is clear from the plot that the maximum amplitude of E will be at most $\mathcal{O}(1/\sqrt{\varepsilon_k})$ as expected and hence the amplitude of the nonlinear acoustics is at most $\mathcal{O}(\varepsilon^2/\delta_L\sqrt{\varepsilon_k})$.
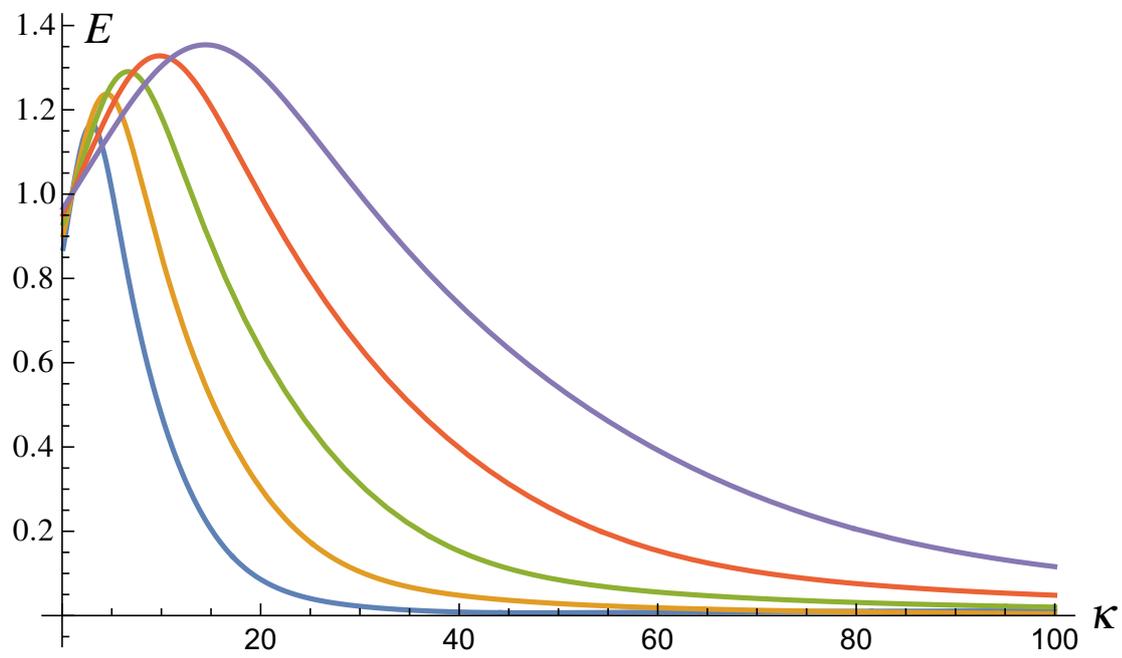
Fig. C.1 $E = \tilde{v}_c / ((y - \kappa)\sqrt{\varepsilon_k})$ against κ for $\varepsilon_k \in \{10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}\}$ showing that E is at most $\mathscr{O}(1/\sqrt{\varepsilon_k})$