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A free-surface-only closure model for linear waves on general deep-water flows

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We present a spatially two-dimensional (2D) set of equations to study the propagation of linear deep-water surface waves over a general steady three-dimensional (3D) background flow with a free surface. No assumptions are made of a flat background free surface or an irrotational background flow. The resulting model is a significant theoretical simplification and enables computations that are orders of magnitude faster than those based on the full linearized Euler equations. The model is obtained by evaluating the linearized Euler equations at the free surface and introducing a closure condition for the vertical derivatives at the free surface. This generalizes a recent result for swirling, axisymmetric background flows (Zuccoli, Brambley & Barkley, 2025, *Phys. Rev. Fluids* 10, 024801). The final model consists of five coupled first-order partial differential equations (PDEs) expressed on the 2D free surface, involving five unknowns: the horizontal and vertical perturbation velocities, the free-surface perturbation height, and the vertical derivative of the perturbation pressure at the surface. Two test problems are used to assess the model: a unidirectional, depth-varying vortical base flow with a flat free surface, and a travelling Gerstner wave. Eigenvalues and eigenfunctions obtained from the surface-only model are compared with those from the full-depth linearized Euler equations. Results show remarkable agreement between the two. While Gerstner waves are known to exhibit three-dimensional short-wavelength instabilities with spanwise dependence, here we find, to the best of our knowledge, the first evidence of a spanwise-constant instability for sufficiently steep Gerstner waves.

Key words: Waves/Free-surface flows → Surface gravity waves; Instability; Interfacial Flows (free surface);

1. Introduction

The propagation of free-surface waves over non-trivial background flows has been extensively studied over the past several decades, motivated both by their rich dynamics and their relevance to a wide range of physical applications. The background flow, or base flow, can profoundly affect the propagation and stability of free-surface waves, leading to phenomena that are absent for a quiescent background flow. Examples include the stability of free surfaces over parallel, unidirectional shear flows (Caponi *et al.* 1991; Longuet-Higgins 1998), the stability of free-surface swirling flows (Acheson 1976; Jansson *et al.* 2006; Tophoj *et al.* 2013; Bach *et al.* 2014; Mougel *et al.* 2014, 2015, 2017; Zuccoli *et al.* 2024), the stability of finite-amplitude periodic gravity waves with constant vorticity (Francius & Kharif 2017), and the modelling of waves generated by wind-driven currents and wakes (Dimas & Triantafyllou 1994; Young & Wolfe 2013).

In this work, we consider linear surface waves in the form of small perturbations to a prescribed base flow. We neglect both viscosity and surface tension. Even in this simple case, the resulting linear free-surface problem is, in general, three-dimensional (3D) in space and therefore frequently computationally demanding. The difficulty is particularly pronounced when the base flow has a deformed free surface and non-zero vorticity, and when the horizontal domain is effectively unbounded. In the latter case, numerical calculations require appropriate treatment of the computational boundaries to prevent artificial reflection of waves (e.g. Zuccoli *et al.* 2024). The simultaneous presence of these features limits the possibility of separating the underlying spatial eigenvalue problem, as is possible in many simpler free-surface flows, and consequently greatly increases the computational cost of obtaining normal-mode solutions.

For potential flows, the three-dimensional free-surface problem can be reduced to a formulation involving quantities defined only on the free surface. This reduction is based on the Dirichlet-to-Neumann (DtN) operator, which maps the value of the velocity potential on the free surface to its normal derivative there, thereby taking into account the fluid interior without explicitly resolving the flow throughout the depth. Following the pioneering work of Zakharov (1968), surface formulations based on the DtN operator have been developed extensively (Craig & Sulem 1993; Milewski & Keller 1996). For linear waves over a quiescent fluid of constant depth H_∞ , the resulting surface equation can be written as

$$\frac{\partial^2 \phi}{\partial t^2} + g\mathcal{G}_0[\phi] = 0, \quad (1.1)$$

where ϕ is the velocity potential of the perturbation at the free surface and $\mathcal{G}_0$ is the DtN operator for a flat surface,

$$\mathcal{G}_0 = (-\nabla_\perp^2)^{1/2} \tanh\left(H_\infty(-\nabla_\perp^2)^{1/2}\right), \quad (1.2)$$

with $\nabla_\perp = (\partial_x, \partial_y)$, where (x, y) are horizontal coordinates. In one horizontal spatial dimension, the operator $\mathcal{G}_0$ is commonly written $\mathcal{G}_0 = -i\partial_x \tanh(-iH_\infty\partial_x)$. In Fourier space, $\mathcal{G}_0$ acts as multiplication by $|\mathbf{k}| \tanh(|\mathbf{k}|H_\infty)$, where $\mathbf{k}$ is the wave vector, thus giving the familiar dispersion relation $\omega^2 = g|\mathbf{k}| \tanh(|\mathbf{k}|H_\infty)$. In the infinite-depth limit, $H_\infty \rightarrow \infty$, the DtN operator reduces to $\mathcal{G}_0 = (-\nabla_\perp^2)^{1/2}$ and the dispersion relation becomes $\omega^2 = g|\mathbf{k}|$.

Torres *et al.* (2018) generalized this approach to study surface waves on an axisymmetric background potential vortex U_0 . They assumed that the background free surface remained flat, thus obtaining

$$\hat{D}_t^2 \phi + g\mathcal{G}_0[\phi] = 0, \quad (1.3)$$

where ϕ is again the velocity potential for linear perturbations, now with respect to a non-zero background flow, $\hat{D}_t = \partial_t + \mathbf{U}_0 \cdot \nabla_\perp$ is the horizontal convective derivative with respect to the background flow $\mathbf{U}_0$, and $\mathcal{G}_0$ is again the general DtN operator defined by (1.2).

The situation is more difficult for waves propagating on vortical background flows. In this case, the classical DtN approach is not applicable and does not provide a closure to obtain a surface-only model. Interior vorticity provides additional degrees of freedom, so that the evolution of perturbation fields cannot in general be determined from their values on the free surface alone, even in the linear case. Generalized DtN formulations exist, e.g. (Castro & Lannes 2015; Ionescu-Kruse & Ivanov 2023), but they do not, in general, yield a closed system involving only free-surface variables because the bulk vorticity remains an independent dynamical variable. Additional assumptions concerning the behaviour of linear surface waves in the vicinity of the free surface are therefore necessary to obtain a closed system involving surface quantities only.

Recently, the authors proposed such a surface-only approach (Zuccoli *et al.* 2025) to study the stability of deep-water surface waves on vortical axisymmetric background flows. The closure was motivated heuristically by the behaviour of linear surface-wave perturbations near the free surface, rather than being derived as a rigorous consequence of the governing equations. The model accounted for both free-surface deformation and the unbounded lateral domain, thus addressing the drawbacks outlined above. Moreover, the model led to four coupled first-order PDEs, which are readily amenable to standard numerical treatment. However, the restriction to axisymmetric background flows severely limits its applicability in practice.

In the present paper, we generalize the reduced surface-only model obtained by Zuccoli *et al.* (2025) to non-axisymmetric, depth-dependent, three-dimensional (3D) base flows. The resulting surface-only equations are derived in §2. To assess the validity and computational efficiency of the surface-only model, we consider two test scenarios of increasing complexity in §3 and §4. For both scenarios, we compare modal solutions of the surface-only equations with modal solutions of the full-depth linearized Euler equations. The first test scenario considers a unidirectional, vertically varying base flow with a periodic structure in the spanwise direction. The corresponding free surface remains flat, but the base flow has non-zero vorticity and gives rise to a non-trivial eigenvalue problem for modal solutions. The second test scenario considers the stability of two-dimensional Gerstner waves. In this case, the base flow varies in the vertical and streamwise directions, has non-zero vorticity, and in some cases has a substantially deformed free surface. For both test scenarios, the surface-only model shows remarkably good agreement with the full-depth calculations in terms of both eigenvalues and eigenfunctions. For the Gerstner waves, the surface-only model also captures the onset of the two-dimensional instability found in the full-depth calculations. Finally, conclusions and opportunities for future research are presented in §5.

2. Mathematical Formulation

2.1. Governing equations and base flow

We assume that viscosity is negligible and hence that the fluid flow is governed by the incompressible Euler equations,

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P - g \mathbf{e}_z, \quad \nabla \cdot \mathbf{U} = 0, \quad (2.1a,b)$$

where $\mathbf{U} = (U, V, W)$ is the velocity, P is the pressure scaled by the constant density ρ , $\mathbf{e}_z$ is a unit vector in the vertical direction, and g is the acceleration due to gravity. The fluid is contained between an upper free surface at $z = H(x, y)$, in general varying with the horizontal spatial coordinates $\mathbf{x}_\perp = (x, y)$, and a bottom boundary at $z = -H_\infty$, where H_∞ is constant. We are in principle interested in the infinite-depth case, $H_\infty = \infty$, but for numerical simulations H_∞ may be finite, in which case the fluid satisfies no penetration through the bottom boundary: $\mathbf{U} \cdot \mathbf{e}_z = 0$ at $z = -H_\infty$. The flow must satisfy kinematic and dynamic boundary conditions at the free surface. We neglect surface tension and assume the fluid above the free surface to be dynamically passive, and in particular, to have a constant pressure $\bar{P}$. Together, these give the boundary conditions

$$P = \bar{P} \quad \text{and} \quad \mathbf{U} \cdot \mathbf{e}_z = \frac{\partial H}{\partial t} + \mathbf{U}_\perp \cdot \nabla_\perp H \quad \text{at} \quad z = H, \quad (2.2)$$

102 where $\mathbf{U}_\perp = (U, V)$ and $\nabla_\perp = (\partial_x, \partial_y)$ are the horizontal velocity and horizontal gradient,
 103 respectively. We consider base flows $\mathbf{U}_0 = (U_0, V_0, W_0)$, P_0 and H_0 satisfying the time-
 104 independent form of equations (2.1) and (2.2).

105 2.2. Linearized equations

We are interested in the behaviour of infinitesimal perturbations to the base flow. Let $\mathbf{u}' = (u', v', w')$ be the perturbation velocity, p' be the perturbation pressure, and $\hat{h}$ the perturbation height, so that the total velocity is $\mathbf{U} = \mathbf{U}_0 + \mathbf{u}'$, the total pressure is $P = P_0 + p'$, and the total fluid elevation is $H = H_0 + \hat{h}$. Here we introduce the notation that $q'(x, y, z, t)$ denotes a perturbation quantity that exists throughout the full depth of the fluid, while $\hat{q}(x, y, t)$ denotes a perturbation quantity on the surface only. Assuming the perturbations to be small, the perturbation dynamics are given by linearizing the incompressible Euler equations (2.1) and boundary conditions (2.2) about the base flow, resulting in

$$D_t \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U}_0 + \nabla p' = 0, \quad (2.3a)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad (2.3b)$$

$$p' = g_0 \hat{h}, \quad \text{on} \quad z = H_0, \quad (2.3c)$$

$$w' = \frac{\partial \hat{h}}{\partial t} + \mathbf{U}_0 \cdot \nabla_\perp \hat{h} + \mathbf{u}' \cdot \nabla_\perp H_0 - \varphi_0 \hat{h}, \quad \text{on} \quad z = H_0, \quad (2.3d)$$

$$w' = 0, \quad \text{on} \quad z = -H_\infty, \quad (2.3e)$$

where $D_t = \partial_t + \mathbf{U}_0 \cdot \nabla$ is the convective derivative due to the base flow. We have defined two quantities on the free surface:

$$g_0 = g + [(\mathbf{U}_0 \cdot \nabla) W_0] \Big|_{H_0}, \quad \varphi_0 = \frac{\partial W_0}{\partial z} \Big|_{H_0} - \frac{\partial \mathbf{U}_{0\perp}}{\partial z} \Big|_{H_0} \cdot \nabla_\perp H_0. \quad (2.4a,b)$$

106 Here, g_0 is the effective gravity accounting for vertical acceleration of the base flow on the
 107 free surface, while φ_0 results from expressing the base normal velocity at the perturbed
 108 free surface $H_0 + \hat{h}$ as an expansion about the base free surface H_0 . Here and throughout,
 109 $|_{H_0}$ means evaluated at the base free surface $z = H_0$. Our interest here is in surface waves
 110 in deep water, for which perturbations decay away from the free surface. For infinite depth,
 111 the bottom boundary condition becomes a requirement that the perturbations decay as
 112 $z \rightarrow -\infty$.

113

2.3. Surface-only model

We seek equations expressed only in terms of quantities defined on the base free surface, and hence quantities that depend only on the horizontal coordinates $\mathbf{x}_\perp$, and not depth z . First, we let $\hat{\mathbf{u}}$ denote the perturbation velocity on free surface,

$$\hat{\mathbf{u}}(\mathbf{x}_\perp, t) = \mathbf{u}'(\mathbf{x}_\perp, H_0(\mathbf{x}_\perp), t), \quad (2.5)$$

114 The components of $\hat{\mathbf{u}}$ are denoted $(\hat{u}, \hat{v}, \hat{w})$ and we use $\hat{\mathbf{u}}_\perp$ to represent the horizontal
115 components $(\hat{u}, \hat{v})$.

We will make use of two identities for derivatives evaluate on the free surface. For any function $f(\mathbf{x}_\perp, z, t)$,

$$(\nabla_\perp f)|_{H_0} = \nabla_\perp(f|_{H_0}) - \nabla_\perp H_0 \frac{\partial f}{\partial z} \Big|_{H_0}, \quad (2.6a)$$

$$\begin{aligned} (D_t f)|_{H_0} &= \frac{\partial(f|_{H_0})}{\partial t} + \mathbf{U}_0|_{H_0} \cdot \nabla_\perp(f|_{H_0}) + \left(W_0|_{H_0} - \mathbf{U}_0|_{H_0} \cdot \nabla_\perp H_0 \right) \frac{\partial f}{\partial z} \Big|_{H_0} \\ &= \hat{D}_t(f|_{H_0}), \end{aligned} \quad (2.6b)$$

116 where we have introduced $\hat{D}_t = \partial_t + \mathbf{U}_0|_{H_0} \cdot \nabla_\perp$ as the convective derivative operator acting
117 on the free surface. Note that the term $(W_0|_{H_0} - \mathbf{U}_0|_{H_0} \cdot \nabla_\perp H_0)$ in (2.6b) is zero due to the
118 kinematic boundary condition (2.2) for the base flow.

Using identities (2.6), the linearized momentum equations (2.3a), the incompressibility constraint (2.3b), and the kinematic boundary condition (2.3d) evaluated on the free surface $z = H_0$ become

$$\hat{D}_t \hat{\mathbf{u}}_\perp + \left[\hat{\mathbf{u}} \cdot \nabla \mathbf{U}_0 \right] \Big|_{H_0} + \nabla_\perp \hat{p} - \nabla_\perp H_0 \frac{\partial \hat{p}'}{\partial z} \Big|_{H_0} = 0, \quad (2.7a)$$

$$\hat{D}_t \hat{w} + \left[\hat{\mathbf{u}} \cdot \nabla W_0 \right] \Big|_{H_0} + \frac{\partial \hat{p}'}{\partial z} \Big|_{H_0} = 0, \quad (2.7b)$$

$$\nabla_\perp \cdot \hat{\mathbf{u}}_\perp + \frac{\partial \hat{w}'}{\partial z} \Big|_{H_0} - \nabla_\perp H_0 \cdot \frac{\partial \hat{\mathbf{u}}'_\perp}{\partial z} \Big|_{H_0} = 0, \quad (2.7c)$$

$$\hat{w} = \hat{D}_t \hat{h} + (\hat{\mathbf{u}}_\perp \cdot \nabla_\perp) H_0 - \varphi_0 \hat{h}, \quad (2.7d)$$

119 where we now explicitly separate the equations for the horizontal and vertical momenta on
120 the free surface, equations (2.7a) and (2.7b).

121 Equations (2.7) are not closed because they contain vertical derivatives at the free surface
122 of full-depth perturbations $\mathbf{u}'$ and p' . Following from our previous work (Zuccoli *et al.*
123 2025), we propose, but do not formally derive, a closure condition to express the vertical
124 velocity gradients that appear in the incompressibility constraint in terms of the time
125 derivative of the vertical pressure gradient.

Equation (2.7d) is the kinematic boundary conditions that the perturbations fields $\hat{\mathbf{u}}'_\perp, w', \hat{h}$ must satisfy on the free surface. Using the dynamic boundary condition (2.3c) to replace $\hat{h}$ by p'/g_0 on the free surface, the boundary condition can be written

$$\Psi[\hat{\mathbf{u}}'_\perp, w', p'](\mathbf{x}_\perp, H_0(\mathbf{x}_\perp), t) = 0, \quad (2.8)$$

where Ψ is the boundary operator

$$\Psi[\hat{\mathbf{u}}'_\perp, w', p'] = w' - (\hat{\mathbf{u}}'_\perp \cdot \nabla_\perp) H_0 - \hat{D}_t \left(\frac{p'}{g_0} \right) + \frac{\varphi_0}{g_0} p'. \quad (2.9)$$

The ansatz we make to close the system is that, for surface gravity waves, the perturbation fields also satisfy

$$\partial_z \Psi[\mathbf{u}'_-, w', p'](\mathbf{x}_\perp, z, t)|_{z=H_0} = 0, \quad (2.10)$$

or equivalently

$$\Psi[\mathbf{u}'_-, w', p'](\mathbf{x}_\perp, H_0(\mathbf{x}_\perp) - \epsilon, t) = o(\epsilon). \quad (2.11)$$

126 In other words, at distance ϵ below the free surface, perturbation fields obey the same
 127 boundary relationship as on the free surface, to within an error that is smaller than ϵ as
 128 $\epsilon \rightarrow 0$. We discuss this assumption below, but first proceed with the manipulations, noting
 129 that everything follows from the ansatz.

From (2.10) we obtain a relationship between vertical gradients of velocity fields and pressure on the free surface,

$$\left. \frac{\partial w'}{\partial z} \right|_{H_0} - \left. \frac{\partial \mathbf{u}'_-}{\partial z} \right|_{H_0} \cdot \nabla_\perp H_0 = \hat{D}_t \left(\left. \frac{1}{g_0} \frac{\partial p'}{\partial z} \right|_{H_0} \right) - \frac{\varphi_0}{g_0} \left. \frac{\partial p'}{\partial z} \right|_{H_0}. \quad (2.12)$$

This is our closure condition. On the left-hand side are the vertical derivatives appearing in the continuity equation on the free surface, (2.7c). On the right-hand side are vertical derivatives of pressure on the free surface, $\partial_z p'|_{H_0}$, and more specifically, there is a time derivative of $\partial_z p'|_{H_0}$. As this unknown will become a dynamic variable to be solved for in our system of equations, and exists only on the surface, we introduce the shorthand notation $\partial_z p'|_{H_0} = \hat{p}_z$. The final system of equations is then obtained by substituting the closure condition (2.12) into the continuity equation (2.7c), and again using the dynamic boundary condition, this time to replace $p'|_{H_0}$ by $g_0 \hat{h}$. This results in a closed system of equations for the surface variables $\hat{\mathbf{u}}_\perp, \hat{w}, \hat{h}, \hat{p}_z$ as functions of $(\mathbf{x}_\perp, t)$:

$$\hat{D}_t \hat{\mathbf{u}}_\perp + (\hat{\mathbf{u}}_\perp, \hat{w}) \cdot [\nabla U_{0\perp}]|_{H_0} + \nabla_\perp (g_0 \hat{h}) - \nabla_\perp H_0 \hat{p}_z = 0, \quad (2.13a)$$

$$\hat{D}_t \hat{w} + (\hat{\mathbf{u}}_\perp, \hat{w}) \cdot [\nabla W_0]|_{H_0} + \hat{p}_z = 0, \quad (2.13b)$$

$$\hat{D}_t (\hat{p}_z / g_0) - \frac{\varphi_0}{g_0} \hat{p}_z + \nabla_\perp \cdot \hat{\mathbf{u}}_\perp = 0, \quad (2.13c)$$

$$\hat{D}_t \hat{h} + \hat{\mathbf{u}}_\perp \cdot \nabla_\perp H_0 - \varphi_0 \hat{h} - \hat{w} = 0. \quad (2.13d)$$

130 We will use the term *surface-only* when referring to equations (2.13) and their solution.
 131 In contrast, we will use the term *full-depth* when referring to the full linearized Euler
 132 equations (2.3) and their solutions.

2.4. Discussion of the surface closure

133
 134 Expressing the linearized Euler equations on the base free surface in terms of free-surface
 135 quantities gives equations (2.7). Vertical derivatives necessarily appear in these equations.
 136 The vertical pressure gradient $\partial_z p'|_{H_0}$ appears in the horizontal momentum equation as a
 137 result of the non-flat free surface, and in the vertical momentum equation through the usual
 138 force balance. Similarly, the surface continuity equation contains vertical derivatives, both
 139 as a consequence of its standard form and as a result of the non-flat free surface. While
 140 none of these vertical derivatives is known from the surface values of pressure or velocity,
 141 the closure condition (2.12) relates the derivatives. In particular, the closure condition
 142 relates the unknown vertical derivatives in the continuity equation to a time derivative of
 143 the free-surface pressure gradient $\partial_z p'|_{H_0}$. This leads to two simultaneous features of the
 144 surface-only model (2.13): the gradient $\partial_z p'|_{H_0} = \hat{p}_z$ becomes an independent dynamical
 145 variable to be solved for, and the continuity equation (2.13c) becomes a time-evolution

equation governing its dynamics. The resulting surface-only model equations are first-order in time with dependent variables $\hat{\mathbf{u}}_\perp$, $\hat{w}$, $\hat{h}$, and also $\hat{p}_z$.

The closure condition is central to the surface-only model and it merits further discussion. For the simplest possible case of linear perturbations about a motionless, infinitely deep base flow, the kinematic and dynamic boundary conditions combine to give $w' = \partial_t p'/g$ on the free surface. Using a velocity potential, it is straightforward to show (see appendix A) that this relationship holds not only on the free surface, but throughout the fluid. Thus, rather trivially, one may differentiate to obtain $\partial_z w' = \partial_t (\partial_z p'/g)$, which is the closure condition (2.12) evaluated in this simplest case.

For the general case of surface gravity waves propagating over vortical base flows, we have been unsuccessful in deriving the closure condition under suitable assumptions other than the ansatz (2.11). We therefore do not claim that the closure condition is exactly valid in general, or even that it holds in some asymptotic sense, other than vanishing vorticity and a flat free surface. The difficulty in deriving such a closure reflects a fundamental difference between potential and vortical flows. In potential flow, the structure of the equations allows the behaviour of the velocity field in the fluid to be determined from data on the free surface through the Dirichlet-to-Neumann map. For a general vortical flow, however, the interior vorticity represents additional bulk degrees of freedom that cannot in general be recovered from free-surface data alone. Consequently, there is no analogous closure based solely on the values of the fields at the free surface without additional assumptions about the interior flow. Indeed, the closure condition (2.12) is only valid for surface waves and does not hold for interior perturbations that span the whole fluid depth; it cannot therefore be derived directly from the governing equations without a suitable surface-waves-only approximation being made. Ansatz (2.11) and the resulting closure condition (2.12) should therefore be understood as a hypotheses specifically about linear surface gravity-wave solutions in the immediate vicinity of the free surface, rather than as a general property of vortical flows or of solutions of the linearized Euler equations.

The authors (Zuccoli *et al.* 2025) previously considered the closure condition in the cylindrical axisymmetric case with no axial flow and showed computationally that the closure condition is well satisfied in deep water for waves on axisymmetric vortices. In the examples considered, the base flows and perturbation fields were both vortical, and the base free surfaces were highly deformed by centrifugal acceleration. In the present work, we extend the closure condition and surface-only model to more general cases. We thus seek to investigate a broader class of examples and to determine under what conditions the closure is approximately satisfied and the surface-only equations produce quantitatively useful solutions.

3. Scenario 1 — waves on a flat shear flow

We validate the surface-only model (2.13) by comparing eigenvalues and eigenmodes from the model with those obtain from the full-depth equations (2.3). In this section, we consider a relatively simple test case in which the base velocity field is unidirectional and the base free surface is flat. Specifically we consider $\mathbf{U}_0(\mathbf{x}_\perp, z) = U_0(y, z)\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector in the x -direction. Irrespective of the form of $U_0(y, z)$, the base free surface is flat, so we set $H_0(\mathbf{x}_\perp) = 0$, and the base pressure field is hydrostatic and given by $P_0(\mathbf{x}_\perp, z) = -gz + \bar{P}$. We examine this relatively simple case because our previous work (Zuccoli *et al.* 2025) did not cover depth-dependent base flows for which there can be a vertical shear at the free surface.

The first base velocity profile we consider is

$$U_0(y, z) = \mathcal{U} \left(1 + \frac{z}{H_\infty} \right) \exp \left\{ \left(\frac{1}{\lambda} - \frac{1}{H_\infty} \right) z \right\} \sin \left(\frac{y}{\lambda} \right), \quad -H_\infty \leq z \leq 0, \quad 0 \leq y \leq 2\pi\lambda. \quad (3.1)$$

where the bottom boundary must satisfy $H_\infty \geq \lambda$. The maximum free-surface velocity is $\mathcal{U}$. The parameter λ is a length scale that controls the magnitude of the shear at the free surface. The maximum magnitudes of the vertical shear, $\partial_z U_0$, and spanwise shear, $\partial_y U_0$, on the free surface are both $\mathcal{U}/\lambda$. For simplicity, we choose to use the same length scale in both directions as rather than introduce a separate length scale for the y direction. This is sufficient for our purposes here, and the example could readily be generalized to allow for different length scales in the two directions.

We non-dimensionalize velocities by $\mathcal{U}$, lengths by λ , and times by $\sqrt{\lambda/g}$, so that $U_0 = \mathcal{U}\widetilde{U}_0$, $(x, y, z) = (\lambda\tilde{x}, \lambda\tilde{y}, \lambda\tilde{z})$, and $t = \sqrt{\lambda/g}\tilde{t}$, with tildes denoting non-dimensional variables. We then drop the tildes for clarity so that the non-dimensional velocity profile becomes

$$U_0(y, z) = \underbrace{\left(1 + \frac{z}{h_\infty} \right) e^{(1-1/h_\infty)z}}_{f_0(z)} \underbrace{\sin(y)}_{q_0(y)}, \quad -h_\infty \leq z \leq 0, \quad 0 \leq y \leq 2\pi, \quad (3.2)$$

where $h_\infty = H_\infty/\lambda$ in the nondimensional depth and where $f_0(z)$ and $q_0(y)$ are functions of z and y defined as indicated. Because we have used an advective velocity scale but a gravitational time scale, the non-dimensionalization of either the full-depth governing equations (2.3) or the surface-only governing equations (2.13) results in the Froude number $\text{Fr} = \mathcal{U}/\sqrt{g\lambda}$ appearing in the usual way as a factor in front of advective terms. In the following, we refer to equations (2.3) and (2.13), but implicitly understand these to mean the non-dimensional forms of these equations. For this flow profile and nondimensionalization, since $W_0 = H_0 = 0$, we have from (2.4) that $g_0 = 1$ and $\varphi_0 = 0$.

3.1. Equations and numerical methods

3.1.1. Full-depth equations

The full-depth equations (2.3) can be entirely expressed in terms of the pressure. This is achieved by taking the divergence of the momentum equation (2.3a), using the continuity equation (2.3b), applying the convective derivative D_t to the equation thus obtained, and finally exploiting the momentum equations for the y and z components, respectively. This yields

$$\left(\frac{\partial}{\partial t} + \text{Fr} f_0(z) q_0(y) \frac{\partial}{\partial x} \right) \nabla^2 p' - 2\text{Fr} f_0(z) q'_0(y) \frac{\partial^2 p'}{\partial x \partial y} - 2\text{Fr} f'_0(z) q_0(y) \frac{\partial^2 p'}{\partial x \partial z} = 0, \quad (3.3a)$$

$$\frac{\partial p'}{\partial z} + \left(\frac{\partial}{\partial t} + \text{Fr} q_0(y) \frac{\partial}{\partial x} \right)^2 p' = 0, \quad \text{on } z = 0, \quad (3.3b)$$

$$\frac{\partial p'}{\partial z} = 0, \quad \text{on } z = -h_\infty, \quad (3.3c)$$

where here f'_0 and q'_0 denote differentiation with respect to a single variable (either y or z) and $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the three-dimensional Laplacian.

Seeking normal modes solutions of the form $p' = \text{Re}(p(y, z)e^{-i\omega t + ikx})$, with k given, equations (3.3) become

$$\begin{aligned} & \left(k\text{Fr} f_0(z)q_0(y) - \omega \right) \left(\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - k^2 p \right) \\ & - 2k\text{Fr} f_0(z)q'_0(y) \frac{\partial p}{\partial y} - 2k\text{Fr} f'_0(z)q_0(y) \frac{\partial p}{\partial z} = 0, \end{aligned} \quad (3.4a)$$

$$\frac{\partial p}{\partial z} - \omega^2 p + 2\omega k\text{Fr}q_0(y)p - k^2\text{Fr}^2q_0^2(y)p = 0, \quad \text{on } z = 0, \quad (3.4b)$$

$$\frac{\partial p}{\partial z} = 0, \quad \text{on } z = -h_\infty, \quad (3.4c)$$

211 subject to periodic boundary conditions in y . System (3.4) gives rise to an eigenvalue
212 problem in the eigenvalues ω and eigenfunctions $p(y, z)$. From $p(y, z)$ the perturbation $\hat{h}$
213 can be obtained via the dynamic boundary condition $\hat{h}(x, y, t) = \text{Re}(p(y, 0)e^{-i\omega t + ikx})$.

214 3.1.2. Surface-only model

We turn now to the surface-only model equations (2.13). Taking the horizontal divergence $\nabla_\perp \cdot$ (2.13a) of the horizontal momentum equations gives

$$\hat{D}_t(\nabla_\perp \cdot \hat{\mathbf{u}}_\perp) + \nabla_\perp^2 \hat{h} + 2\text{Fr}f_0(0)q'_0(y) \frac{\partial \hat{v}}{\partial x} + \text{Fr}f'_0(0)q_0(y) \frac{\partial \hat{w}}{\partial x} = 0. \quad (3.5)$$

We now substitute $\nabla_\perp \cdot \hat{\mathbf{u}}_\perp = -\hat{D}_t \hat{p}_z$ from (2.13c), $\hat{p}_z = -\hat{D}_t \hat{w}$ from (2.13b), and $\hat{w} = \hat{D}_t \hat{h}$ from (2.13d), to arrive at

$$\hat{D}_t^4 \hat{h} + \nabla_\perp^2 \hat{h} + 2\text{Fr}f_0(0)q'_0(y) \frac{\partial \hat{v}}{\partial x} + \text{Fr}f'_0(0)q_0(y) \frac{\partial}{\partial x} \hat{D}_t \hat{h} = 0. \quad (3.6)$$

Finally, applying $\hat{D}_t$ to (3.6) and substituting $\hat{D}_t \hat{v} = -\partial_y \hat{h}$ from the y -component of (2.13a) yields the following PDE in the free surface height $\hat{h}(x, y, t)$:

$$\hat{D}_t^5 \hat{h} + \hat{D}_t \nabla_\perp^2 \hat{h} - 2\text{Fr}f_0(0)q'_0(y) \frac{\partial^2 \hat{h}}{\partial x \partial y} + \text{Fr}f'_0(0)q_0(y) \frac{\partial}{\partial x} \hat{D}_t^2 \hat{h} = 0. \quad (3.7)$$

For the base flow under consideration, $f_0(0) = 1$, and the convective derivative acting on the free surface is therefore $\hat{D}_t = \partial_t + \text{Fr}q_0(y)\partial_x$. Seeking normal modes of the form $\hat{h}(x, y, t) = \text{Re}(h(y)e^{-i\omega t + ikx})$, equation (3.7) yields the eigenvalue problem

$$\begin{aligned} & \left(k\text{Fr} q_0(y) - \omega \right)^5 h + \left(k\text{Fr} q_0(y) - \omega \right) (h'' - k^2 h) \\ & - 2k\text{Fr} q'_0(y) h' - k\text{Fr} f'_0(0)q_0(y) \left(k\text{Fr} q_0(y) - \omega \right)^2 h = 0, \end{aligned} \quad (3.8)$$

215 subject to periodic boundary conditions in y . The term containing f'_0 is of particular interest
216 because it accounts for the effect of the base vertical shear on the eigenvalue problem.

217 3.1.3. Numerical method

We solve equations (3.4) and (3.8) numerically by means of a spectral collocation method (Trefethen 2000), using Fourier modes along the spanwise direction y , and for (3.4), Chebyshev modes along the vertical direction z . This involves, for the numerical solution of the exact stability problem only, re-mapping the vertical domain from $(-h_\infty, 0) \rightarrow (-1, 1)$

when the depth is finite, and re-mapping $(-\infty, 0) \rightarrow (-1, 1)$ when the depth is infinite. For the latter, we employ the following map (Boyd 2001)

$$z = B_{\text{map}} \tan \left[\frac{\pi}{4} (z_c - 1) \right], \quad \text{with } z \in (-\infty, 0], \quad \text{and } z_c \in [-1, 1], \quad (3.9)$$

where B_{map} is a tuning parameter which sets the clustering of points close to the free surface $z = 0$. We use spatial discretizations $(N_y, N_z) = (52, 100)$ along the spanwise and vertical directions respectively, and use $B_{\text{map}} = 4$. Further details on the numerical discretization and convergence of the eigensolutions for this first validation problem are contained in appendix B. It is worth noting how much faster the computations using the surface-only model equations are compared to the full-depth counterpart. For this example, we find mean execution times of $T_{\text{mean}}^{(FD)} = 6 \times 10^3$ s for the full-depth Euler equations, while $T_{\text{mean}}^{(SO)} = 8 \times 10^{-2}$ s for the surface-model equations, showing that the model is many orders of magnitude faster than the full-depth equations.

Finally, an important aspect of the surface-only model is that its spectrum comprises in general both “physical” and “unphysical” eigenvalues. This feature arises because the surface-only model equations intrinsically cannot distinguish whether the fluid extends infinitely below or infinitely above the base free surface. Since our goal is to compute modes which decay as $z \rightarrow -\infty$, in order to remove the unphysical eigenvalues we define a proxy β for the average vertical decay rate. If β is sufficiently positive, then that mode is retained; if not, it is discarded. For this scenario, we use the criteria

$$\beta = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left(\frac{\partial_z p'|_{H_0}}{p'|_{H_0}} \right) dy = \omega^2 + k^2 \text{Fr}^2 / 2 > \delta_\beta, \quad (3.10)$$

where $\delta_\beta = 0.01$ is the threshold chosen for the results that follow. Further details are contained in appendix C.

3.2. Results

We now compare eigenvalues and eigenmodes from the surface-only model (3.8) to those from the full-depth equations (3.4). We also assess directly whether the closure condition (2.12) is satisfied by the full-depth solutions. Throughout this section we fix the streamwise wavenumber at $k = 1$, corresponding to eigenmodes whose streamwise wavelength is comparable to the length scales of the base flow.

In figure 1 we show the first five positive eigenvalues as a function of h_∞ from the full-depth (full lines) and surface-only (points) equations. The Froude number is $\text{Fr} = 0.5$. All eigenvalues are real. We plot only the positive half of the spectrum, since the spectrum is symmetric about zero (discussed further below). It can be seen that the surface-only eigenvalues not only accurately match those from the exact numerical calculations in the deep-water limit ($h_\infty \rightarrow \infty$), but also for h_∞ of order unity. Significant deviations between the surface-only and full-depth cases begin at h_∞ between 1 and 2. As the fluid depth becomes shallow, $h_\infty < 1$, the deviations become large.

To gain further insight into the effect of fluid depth, we plot in figure 2 (left panels) pressure contours of the first eigenfunction from the full-depth computations at four values of dimensionless fluid depth $h_\infty \in \{0.5, 1, 2, \infty\}$. The eigenmodes at $h_\infty = 1$ and $h_\infty = 0.5$ are clearly influenced by the bottom boundary, and unsurprisingly, these cases are not in the deep-water regime and the model is not expected to hold.

Additionally, we assess directly whether the closure condition (2.12) holds for the exact full-depth eigenmodes. For the case under consideration, the dimensionless form of the

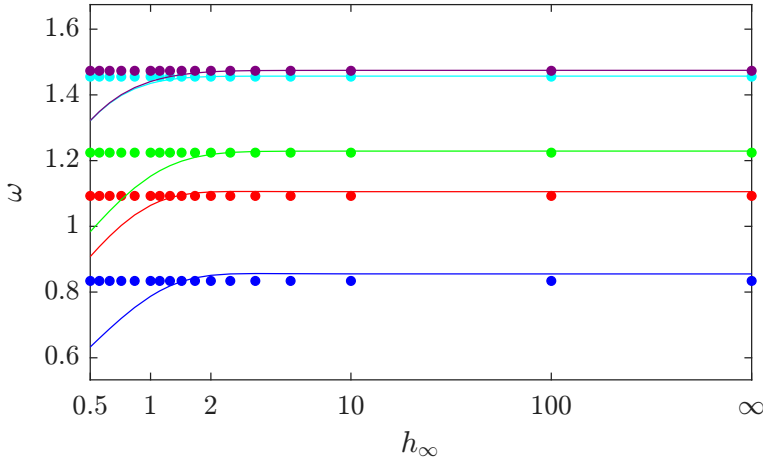


Figure 1. Trend of the first five positive eigenvalues as a function of dimensionless depth h_∞ for $Fr = 0.5$. The base flow velocity profile is given by equation (3.2). Eigenvalues from full-depth equations, viewed as exact, are shown in full lines. Eigenvalues from the surface-only model are shown as points.

closure equation reduces to

$$\left. \frac{\partial w'}{\partial z} \right|_{H_0} = \hat{D}_t \left(\left. \frac{\partial p'}{\partial z} \right|_{H_0} \right), \quad (3.11)$$

where now $\hat{D}_t = -i\omega + ikq_0(y)$ from the normal modes representation. The left-hand and right-hand sides of this equation are plotted in figure 2 (right panels). At $h_\infty = \infty$ and $h_\infty = 2$ there is close agreement between the curves, implying that the model is accurately closed in these cases. The agreement is not perfect, however, even at $h_\infty = \infty$. For $h_\infty = 1$ and $h_\infty = 0.5$ the closure condition is clearly failing and hence one could not expect the model to produce reliable results.

We continue by comparing the surface-only eigenmodes to the eigenmodes from the full-depth equations. Figure 3 plots the first five eigenmodes in terms of the free-surface elevation h for $Fr \in \{0.25, 0.5, 1\}$, with $h_\infty = \infty$. All eigenmodes are normalized to have an L_2 norm of 1, the normalization convention used throughout this work. While differences are visible between the two sets of modes, overall the surface-only eigenmodes capture very well the corresponding eigenmodes from the full-depth equations. It is evident that the eigenmodes are not symmetric under reflection about $y = \pi$ or, equivalently, taking into account the periodic domain, about $y = 0$. It is straightforward to show from the symmetry of the base flow that if $h(y)$ is an eigenmode with real frequency ω , then $h(-y)$ is also an eigenmode with frequency $-\omega$. Hence, the negative half of the spectrum, not shown in figure 1, corresponds to the reflected versions of the modes shown in figure 3. Since the eigenvalues are real, the time-dependent modes can be interpreted as travelling waves in the streamwise direction of the form $h(y) \cos(kx - \omega t)$. The modes shown in figure 3 corresponds to waves travelling in the $+\mathbf{e}_x$ direction, since $k = 1$, $\omega > 0$.

Finally, we address the quantitative accuracy of the surface-only eigenvalues compared with those from the full-depth equations. Let $\omega_i^{(FD)}$ and $\omega_i^{(SO)}$ denote, respectively, the full-depth and surface-only eigenvalues. In figure 4 we plot the relative error between the two $|\omega_i^{(FD)} - \omega_i^{(SO)}|/|\omega_i^{(FD)}|$ for the first ten eigenvalues computed at five values of Froude number $Fr \in \{0, 0.01, 0.25, 0.5, 1\}$. The error is below 2% over the entire range of modes considered, confirming the accuracy and robustness of the surface-only model in

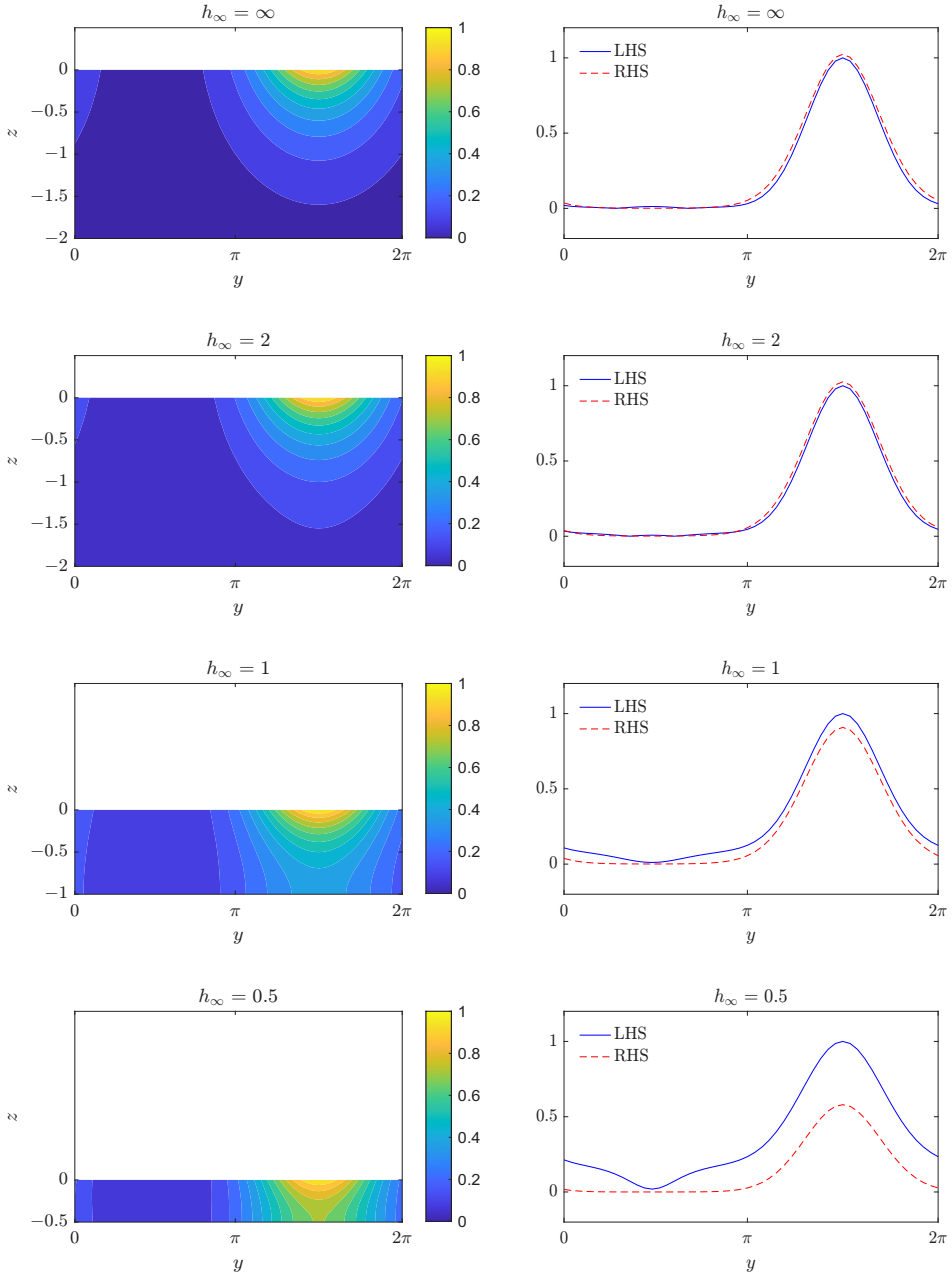


Figure 2. Accuracy of the closure condition (3.11) for the first surface-wave modes with Froude number $Fr = 0.5$ at four values of dimensionless fluid depth h_∞ . Left panels illustrate a snapshot of the modes at time $t = 0$ with contours of the pressure field $\text{Re}(p(y, z))$. The modulus of both left-hand side (LHS) and right-hand side (RHS) of the closure condition (3.11) are evaluated and plotted in the right panels, with the eigenfunction normalized so that the maximum value of the left-hand side is 1.

274 these cases. For $Fr = 0$, the agreement is at the level of numerical precision because the
 275 base flow is motionless, and hence both the full stability equations and the surface-only

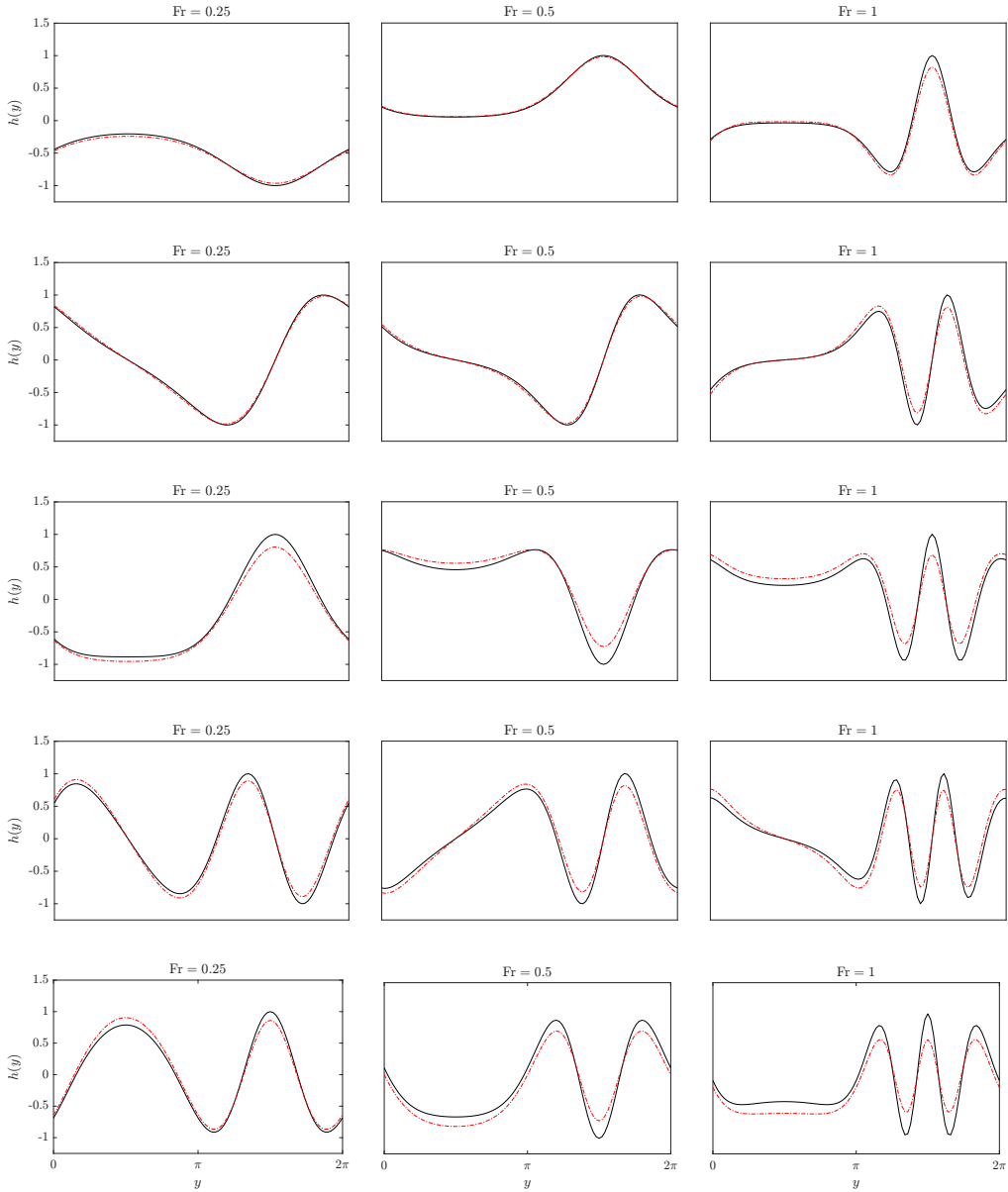


Figure 3. Comparison of the surface-only and full-depth eigenfunctions for Froude numbers $Fr \in \{0.25, 0.5, 1\}$, with $h_\infty = \infty$. The free-surface elevation $h(y)$ is plotted for the first five modes, corresponding to the five eigenvalues seen in figure 1. The panels are ordered by frequency with the top panel the first (lowest-frequency) mode. Full-depth modes are shown in full black lines and surface-only modes dashed red lines.

276 model admit the same solutions, with exact eigenvalues given by $\omega_n = (n^2 + k^2)^{1/4}$, $n \in \mathbb{Z}$
 277 (see appendix D for further details).

278 3.3. Velocity profile with depth-dependent surface shear

For the base velocity profile studied thus far, the surface-velocity gradients $\partial_z U_0$ and $\partial_y U_0$ are independent of the fluid depth. This is why the surface-only eigenvalues in figure 1 do not vary with depth h_∞ ; the surface-only model is influenced by the shear of the base flow

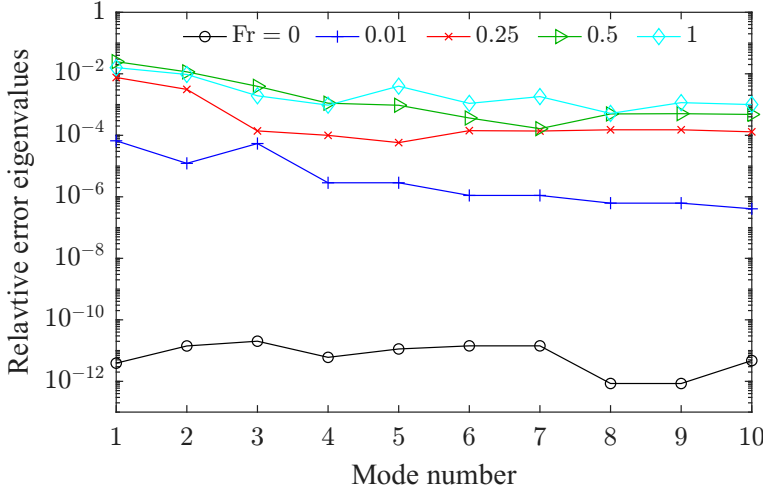


Figure 4. Relative error of the first ten eigenvalues between full-depth and surface-only models for different Froude numbers.

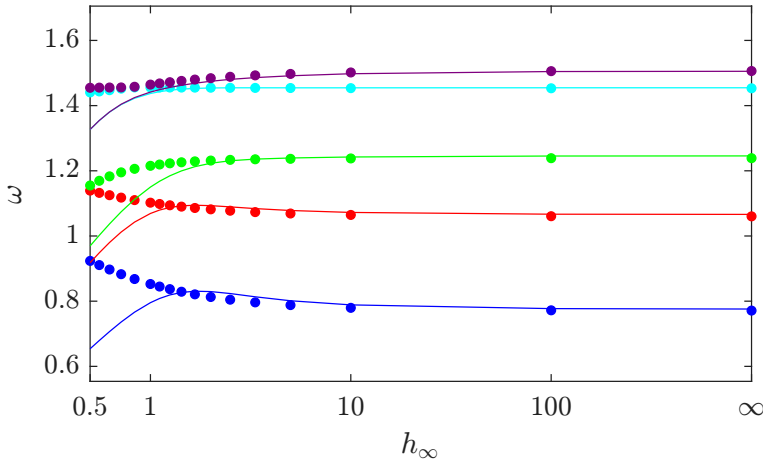


Figure 5. Trend of the first five positive eigenvalues as function of dimensionless depth h_∞ for $Fr = 0.5$. The base flow velocity profile is given by equation (3.12). Eigenvalues from the full-depth equations, viewed as exact, are shown in full lines. Eigenvalues from the surface-only model are shown as points.

at the free surface, and not fluid depth directly. However, it is also natural for the vertical shear of a fluid flow to increase as the fluid depth decreases, and so we now briefly consider such a case using a variant of the base velocity profile studied thus far. Rather than using $f_0(z)$ appearing (3.2), we set

$$f_0(z) = \frac{\sinh(1 + z/h_\infty)}{\sinh(1)}. \quad (3.12)$$

279 The base flow is otherwise unchanged from the previous test case. For the surface-only
 280 eigenvalue problem (3.8), the only change is in f'_0 at the free surface. Previously this
 281 was $f'_0(0) = 1$, while now it is $f'_0(0) = \coth(1)/h_\infty$, so that the fluid depth enters the
 282 surface-only equations.

In figure 5 we show the first five positive eigenvalues as function of h_∞ from the full-depth equations (full lines) and the surface-only model (points). The Froude number is $Fr = 0.5$. Comparing figure 5 with figure 1, we see that in both cases there is good agreement between the surface-only model and full-depth eigenvalues down to fluid depths h_∞ of order unity. However, unlike in figure 1, in figure 5 we see that eigenvalues begin to depend depth when h_∞ is of order 10. This is due to the variation of the base shear with depth. The surface-only model clearly picks up the initial deviations of the eigenvalues from the infinite-depth limit as h_∞ decreases. Once the fluid depth becomes shallow, $h_\infty < 1$, the surface-only model fails and the deviations become large. While not shown, a direct validation of the closure condition for this profile leads to results very similar to what is shown in figure 2. Likewise, a comparison of eigenfunctions strongly resembles what is seen in figure 3.

4. Scenario 2 — waves on a Gerstner wave

Gerstner waves (Gerstner 1809; Rankine 1863), also known as trochoidal waves, are exact solutions to the Euler equations with both depth-dependent velocity and free-surface deformation, and as such these waves provide an interesting and important class of base flows to test our surface-only model. Gerstner-waves are typically specified in Lagrangian coordinates, and are valid in the infinitely deep-water regime. When considered in a reference frame moving with the wave speed, these provide exact steady base flows with nonzero vorticity distributions. In the following, we compute linear surface wave perturbations propagating on Gerstner wave base flows using our surface-only model and compare them with corresponding surface waves from full-depth linearized Euler calculations.

4.1. The base flow

Let (x, z) be Eulerian spatial coordinates in a frame moving with the wave, let $(\mathcal{X}, \mathcal{Z})$ be Lagrangian coordinate in a fixed reference frame, and let (a, b) be the Lagrangian coordinates in the reference frame moving with the wave, i.e. $a = \mathcal{X} + ct$, $b = \mathcal{Z}$, with $c = \sqrt{g/k}$ denoting the phase speed of the wave in deep-water, and $2\pi/k$ its wavelength. We define the Euler–Lagrange map between the two coordinate systems,

$$x = a + \frac{1}{k} e^{kb} \sin(ka), \quad z = b - \frac{1}{k} e^{kb} \cos(ka), \quad (4.1a,b)$$

for $b \in (-\infty, b_0]$, where $b = b_0 \leq 0$ dictates the location of the steady free surface. The corresponding velocity components in the steady Lagrangian reference frame are given by

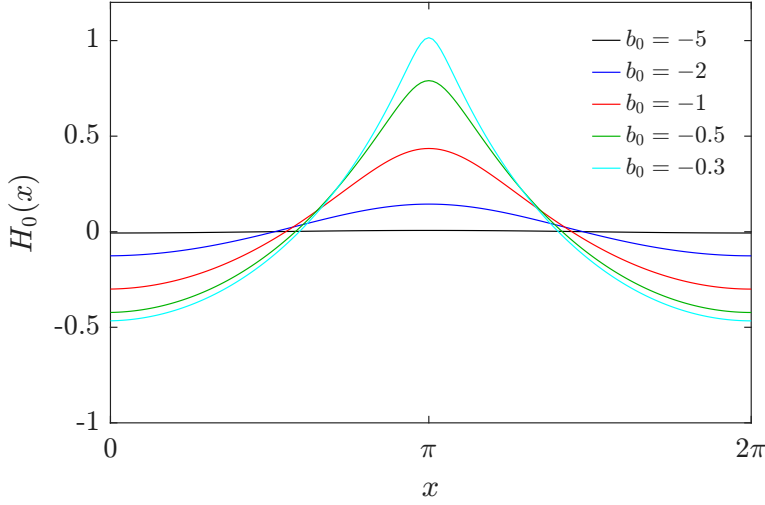
$$U_0 = c + c e^{kb} \cos(ka), \quad W_0 = c e^{kb} \sin(ka). \quad (4.2a,b)$$

The base free surface $z = H_0(x)$ is determined implicitly by fixing $b = b_0$ and varying a in the Euler-Lagrange map (4.1).

We non-dimensionalize lengths by $1/k$, so that $a = \tilde{a}/k$, $x = \tilde{x}/k$, etc. (with tildes denoting nondimensional variables), and times by $1/\sqrt{gk}$, so that $\tilde{c} = 1$. We now drop the tildes for clarity. Additionally, we shift the coordinate z vertically upward by $-b_0$, so that the mean level of the free surface $H_0(x)$ is approximately at $z = 0$ for any value of b_0 . The dimensionless base flow is then given by

$$x = a + e^b \sin(a), \quad U_0 = 1 + e^b \cos(a) \quad (4.3a,b)$$

$$z = b - b_0 - e^b \cos(a), \quad W_0 = e^b \sin(a), \quad (4.3c,d)$$


 Figure 6. Base free surfaces $H_0(x)$ for representative values of b_0 .

308 where $a \in [0, 2\pi]$ and $b \in [-\infty, b_0]$. The base flows are thus parameterized by the
 309 single parameter $b_0 < 0$. In the limit $b_0 \rightarrow -\infty$ the base velocity field becomes uniform
 310 in the horizontal direction and the free surface flat. For increasing values of b_0 the free
 311 surface deformation becomes more significant, with the limiting case of $b_0 = 0$ giving a
 312 cusp-shaped free surface. Examples of the free surface shapes are plotted in figure 6.

313

4.2. Perturbations dynamics

The Gerstner base flows are two-dimensional, and we will only considered perturbations
 that are also two dimensional, with velocity field given by $\mathbf{u}' = (u', w')$. Note that Gerstner
 waves are known to be unstable to three-dimensional perturbations for $b_0 > -\ln 3$ (Leblanc
 2004), although this short-wavelength instability requires three dimensions and so is absent
 in the two-dimensional case considered here. (This can be seen by choosing a mode wave
 vector in the x - z plane in equation (3.3) in Leblanc 2004.) Also unlike Leblanc (2004), we
 do not require a short-wavelength perturbation, and instead take the perturbations to be
 characterized by the same spatial scale $1/k$ and time scale $1/\sqrt{gk}$ as for the base flow. The
 exact two-dimensional perturbations equations (2.3) in dimensionless form then read

$$\frac{\partial u'}{\partial t} + \mathbf{U}_0 \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U}_0 + \frac{\partial p'}{\partial x} = 0, \quad (4.4a)$$

$$\frac{\partial w'}{\partial t} + \mathbf{U}_0 \cdot \nabla \mathbf{w}' + \mathbf{u}' \cdot \nabla \mathbf{W}_0 + \frac{\partial p'}{\partial z} = 0, \quad (4.4b)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \quad (4.4c)$$

$$\frac{\partial p'}{\partial t} = g_0 \left(w' - \frac{dH_0}{dx} u' \right) - U_0 \left[\frac{\partial p'}{\partial x} + \frac{dH_0}{dx} \frac{\partial p'}{\partial z} - \frac{1}{g_0} \frac{dg_0}{dx} p' \right] + \varphi_0 p' \quad \text{on } z = H_0. \quad (4.4d)$$

where we have substituted the dynamic boundary condition (2.3c) into the kinematic
 boundary (2.3d), to obtain a single boundary condition in p . The quantities $g_0(x)$ and

$\varphi_0(x)$ are given by

$$g_0 = \left[1 + U_0 \frac{\partial W_0}{\partial x} + W_0 \frac{\partial W_0}{\partial z} \right] \bigg|_{z=H_0}, \quad \text{and} \quad \varphi_0 = \left[\frac{\partial W_0}{\partial z} - \frac{dH_0}{dx} \frac{\partial U_0}{\partial z} \right] \bigg|_{z=H_0}. \quad (4.5a,b)$$

Additionally, periodic boundary conditions are imposed in the x -direction, identifying $x = 2\pi$ with $x = 0$. While one could consider the perturbation problem over multiple wavelengths of the base flow, for example by using Floquet theory, for simplicity we do not pursue that further here.

4.2.1. Numerical considerations

As the upper boundary of the domain is given by $H_0(x)$, which is not constant, it is numerically convenient to introduce the shifted vertical coordinate $\xi = z - H_0(x) \in (-\infty, 0]$. The derivatives transform into $\partial_x \rightarrow \partial_x - (dH_0/dx)\partial_\xi$ and $\partial_z \rightarrow \partial_\xi$, so that equations (4.4) are re-written accordingly in terms of (x, ξ, t) . In order to discretize the equations along the vertical direction despite the infinite depth, we use the same map (3.9) as for the previous scenario. Finally, we decompose the unknowns into normal modes of the form $[u', w', p'] = \text{Re}([u(x, \xi), w(x, \xi), p(x, \xi)]e^{-i\omega t}) = \text{Re}(\mathbf{w}(x, \xi)e^{-i\omega t})$. Note that, since $[u', w', p']$ are real, if $(\mathbf{w}, \omega)$ is a solution then so too is $(\mathbf{w}^*, -\omega^*)$, where $*$ denotes the complex conjugate. This yields the following generalized eigenvalue problem in the eigenvalues ω

$$\mathcal{A}(\partial_x, \partial_\xi; b_0)\mathbf{w} = \omega\mathcal{B}(b_0)\mathbf{w}. \quad (4.6)$$

Problem (4.6) is solved numerically in primitive variables using a collocation spectral method, similar to the one employed in section 3 and described in appendix B.

4.2.2. The surface-only model

The surface-only model equations for this case reduce to

$$\hat{D}_t \hat{u} + \hat{u} \partial_x U_0|_{H_0} + \hat{w} \partial_z U_0|_{H_0} + \partial_x (g_0 \hat{h}) - H'_0 \hat{p}_z = 0, \quad (4.7a)$$

$$\hat{D}_t \hat{w} + \hat{u} \partial_x W_0|_{H_0} + \hat{w} \partial_z W_0|_{H_0} + \hat{p}_z = 0, \quad (4.7b)$$

$$\hat{D}_t (\hat{p}_z / g_0) - \frac{\varphi_0}{g_0} \hat{p}_z + \partial_x \hat{u} = 0, \quad (4.7c)$$

$$\hat{D}_t \hat{h} + \hat{u} H'_0 - \varphi_0 \hat{h} - \hat{w} = 0, \quad (4.7d)$$

where $\hat{D}_t = (\partial_t + U_0|_{H_0}\partial_x)$. This is a system of four PDEs in the four primitive variables $(\hat{u}, \hat{w}, \hat{h}, \hat{p}_z)$. Seeking normal modes of the form $e^{-i\omega t}$ results in an eigenvalue problem. The spatial domain of this eigenvalue problem is simply the periodic interval $x \in [0, 2\pi]$, and so is easily solved numerically. As for the numerical implementation of the first scenario in section 3.1, the surface-model equations (4.7) give rise to some unphysical eigenvalues which need to be discarded accordingly. Details can be found in appendix C.

4.3. Results

We compare eigensolutions obtained from the full-depth linearized Euler equations (4.4) with those coming from the surface-only model equations (4.7) for various free-surface deformations of the base solution, corresponding to different values of b_0 . We begin with a comparison of eigenvalues shown in figure 7(a). All eigenvalues shown in the figure are real, corresponding to neutrally-stable propagating modes. There are also unstable modes discussed in §4.4. We index the eigenvalues such that $\dots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \dots$. For $b_0 = -5$, the peak-to-peak variation in the free-surface elevation of the base flow is

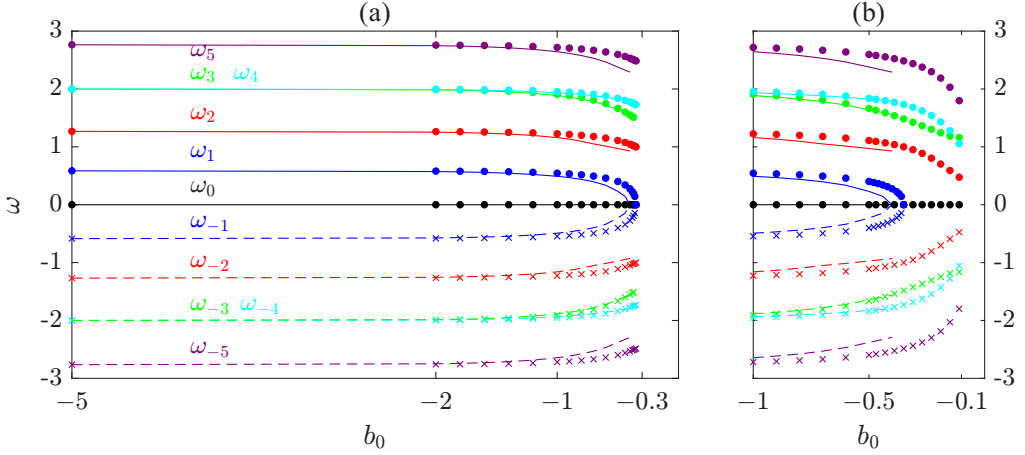


Figure 7. (a) Comparison of the eigenvalues from the full-depth linearized Euler stability problem (solid lines) and the surface-only model equations (markers) up to $b_0 = -0.3$. (b) Close-up of the eigenvalue branches for the surface-model equations for values of b_0 up to -0.1 , where the peak of the base free surface is close to forming a cusp.

$O(10^{-2})$, which is already close to the flat case corresponding to the limit $b_0 \rightarrow -\infty$. Accordingly, each eigenvalue ω_j tends to a closed-form solution valid in this limit, $\omega_j = n \pm \sqrt{|n|}$ for some $n \in \mathbb{Z} \setminus \{0\}$ (see appendix D.2). As b_0 is increased, corresponding to larger base-flow free-surface amplitudes, the surface-only eigenvalues follow closely the full-depth eigenvalues up to $b_0 \simeq -1$. Beyond this point, the discrepancy between the two eigenvalue branches becomes more noticeable, although the surface-only model still qualitatively captures the behaviour of the full-depth linearized Euler eigenvalues. In particular, it is worth noting that the first non-zero eigenvalues, $\omega_{\pm 1}$, collapse to zero at a finite value $b_0 = -0.43$ for the full-depth linearized Euler system and $b_0 = -0.35$ for the surface-only model. The eigenvalues then split into pairs of complex-conjugate eigenvalues with zero real part, resulting in an instability of the system described further in §4.4.

To further validate the accuracy of our surface-only model, we plot in figure 8 the perturbation height $h(x)$ of the eigenmodes corresponding to the eigenvalues $\omega_0, \dots, \omega_5$. The phase and amplitude of each eigenmode have been chosen to make the real and imaginary parts of the eigenfunction as independent as possible, and to give each eigenmode an L_2 norm of 1. As with the eigenvalues, the agreement between the two sets of eigenmodes is remarkably good. Even for the substantially deformed free surface at $b_0 = -0.5$, where visible differences appear between the two sets of eigenfunctions, the surface-only eigenfunctions still capture the features of the full-depth eigenfunctions very well.

The eigenmode corresponding to $\omega_0 = 0$ is a Goldstone mode (Goldstone 1961), arising from the broken translational symmetry of the base flow. While the governing equations are translationally invariant, the steady base flow breaks this continuous symmetry. An infinitesimal translation of the base flow generates another physically equivalent steady base state, and hence the corresponding infinitesimal perturbation is neutrally stable. In our case, the Goldstone mode has eigenvalue $\omega_0 = 0$, with eigenmode given by $(u', w', p', \hat{h}) = \partial_x(U_0, W_0, P_0, H_0)$

Due to large deformations of the base state as b_0 approaches zero, we are able to compute eigenmodes from the full-depth stability equations reliably only up to $b_0 = -0.3$. However, we can then use our reduced surface-only model to predict the trend of the eigenvalues as b_0 gets closer to zero, where a cusp-shaped base free surface forms (the maximum value of

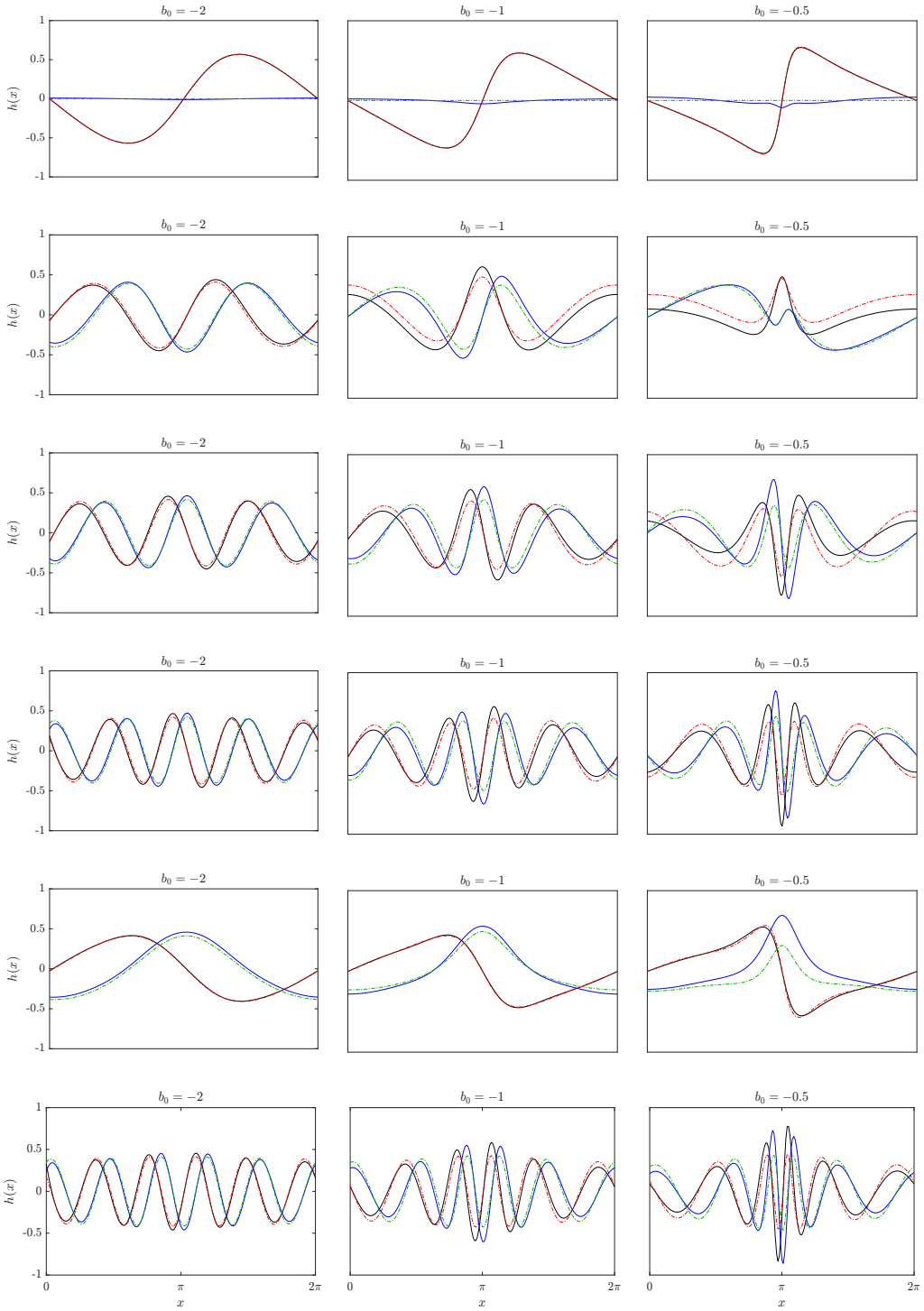


Figure 8. Comparison of full-depth linearized Euler and surface-only model eigenfunctions for $b_0 = -2, -1, -0.5$. Plotted are the real and imaginary parts of the free surface elevation $h(x)$. Real part Euler (black line); real part model (red dashed line); imaginary part Euler (blue line); imaginary part model (green dotted line). Rows 1–6 show modes 0–5. An animation of the results for $b_0 = -0.5$ is given in Movie 1 in the supplementary material.

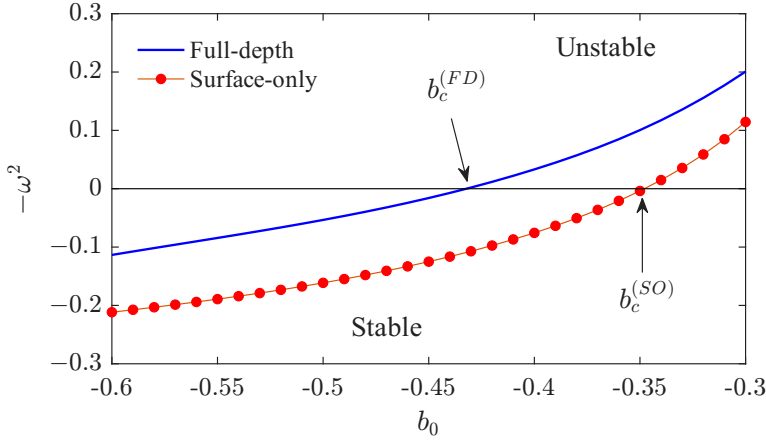


Figure 9. Instability diagram as function of b_0 . The critical values of b_0 are $b_c^{(FD)} = -0.43$ for the full-depth equations and $b_c^{(SO)} = -0.35$ for the surface-only model. For $b_0 < b_c$, the system is stable; for $b_0 > b_c$, the system is unstable.

b_0 considered here being $b_0 = -0.1$). The continuation of the eigenvalues branches within this extreme parameter limit is shown in panel (b) of figure 7.

4.4. Onset of instability

The lowest-frequency modes $\omega_{\pm 1}$ become unstable at the critical values $b_0 = b_c^{(FD)} = -0.43$ for the full-depth equations and $b_0 = b_c^{(SO)} = -0.35$ for the surface-only model equations. In figure 9, we show the onset of the instability by plotting $-\omega_{\pm 1}^2$ as a function of b_0 . When $-\omega_{\pm 1}^2 < 0$, the eigenvalues $\omega_{\pm 1}$ form a real pair, corresponding to neutrally-stable oscillations. When $-\omega_{\pm 1}^2 > 0$, the eigenvalues form a purely imaginary pair; we label the eigenvalue with positive imaginary part as $\omega_1 = i\sigma_1$, where $\sigma_1 > 0$, corresponding to exponential growth of the form $e^{\sigma_1 t}$.

This scenario is characteristic of a Hamiltonian pitchfork bifurcation (Marsden 1992). There are two caveats concerning the interpretation of this bifurcation. First, we consider here only linear stability and do not investigate the nonlinear states that may emerge following the instability. We therefore do not claim to establish a Hamiltonian pitchfork bifurcation in the full nonlinear sense. Rather, we observe that the eigenvalues exhibit the standard collision scenario associated with a Hamiltonian pitchfork bifurcation. Second, at the critical point the persistent zero eigenvalue associated with the Goldstone mode coincides with the pair of eigenvalues $\omega_{\pm 1}$ that become unstable. The eigenmode corresponding to the stable wave increasingly resembles the Goldstone mode as the critical point is approached: its frequency tends to zero, and at the critical point the real part of the eigenfunction coincides with the Goldstone mode. Thus, at the critical point, the zero eigenvalue has algebraic multiplicity at least three but geometric multiplicity one. The linearization at the critical point therefore has a 3×3 Jordan block associated with the zero eigenvalue.

In figure 10, we show the unstable eigenfunctions for both the full-depth and surface-only equations slightly after the onset of the instability. The two eigenmodes differ, but they have similar characteristics. Both resemble the Goldstone mode to some degree, although the Goldstone mode is antisymmetric about $x = \pi$, whereas neither of the unstable modes is. The unstable mode illustrates a small issue with the surface-only model. Let $\bar{h}$ be the

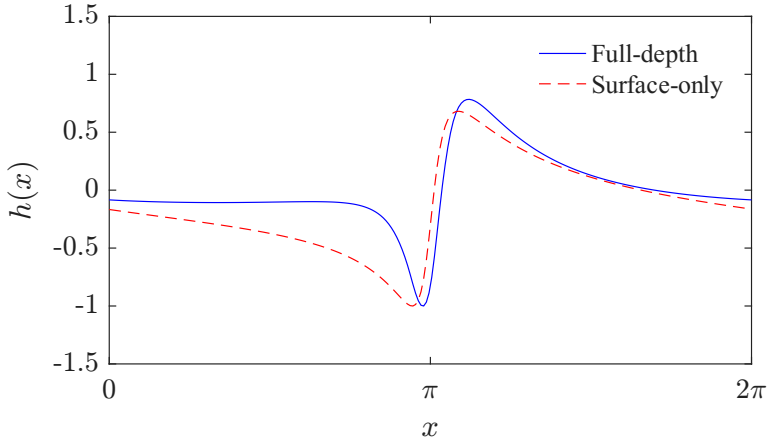


Figure 10. Comparison of the unstable eigenfunction between full-depth and surface-only model solutions computed at the same reduced bifurcation parameter $(b_0 - b_c)/|b_c| = 0.0698$, corresponding to $b_0 = -0.4$ for full-depth and $b_0 = -0.32$ for surface-model.

mean of h over the horizontal domain. There is no constraint in the surface-only equations requiring $\bar{h} = 0$, as is necessary for conservation of mass. For all the stable solutions considered up to this point, $\bar{h}$ is found to be zero. However, it is evident in figure 10 that $\bar{h} < 0$ for the plotted surface-only eigenmode. Nonetheless, the surface-only eigenfunction is qualitatively similar to the full-depth eigenfunction even in this case, with the full-depth eigenfunction having $\bar{h}$ indistinguishable from zero.

Figures 7–10 demonstrate that the surface-only model reproduces very well the behaviour of the full-depth calculations as b_0 is varied, both in terms of the eigenvalues and the eigenfunctions. Significantly, it successfully captures the onset of the instability. This provides strong evidence that the surface-only model is capable of reproducing dynamics of linear surface waves on a non-trivial base flow with substantial vorticity and a significantly deformed free surface.

5. Conclusions

We have generalized the spatially two-dimensional set of equations previously derived in Zuccoli *et al.* (2025) to study the propagation of deep-water surface waves over a general steady base flow with a free surface. The model reduces the full-depth 3D problem to a 2D problem in horizontal coordinates only, which enables the computation of surface waves propagating on general depth-dependent flows (including non-potential flows) whose steady base free surface is not flat. The surface-only model is obtained by evaluating the linearized Euler equations on the base free surface and then introducing a closure condition to account for the vertical derivatives at the free surface. These surface-only equations have been derived from a heuristic argument following the same idea introduced in Zuccoli *et al.* (2025).

We have verified the validity and the accuracy of the reduced surface-model to capture deep-water surface-wave modes for two test scenarios of increasing complexity. The first test has been performed against a unidirectional, depth-dependent base flow of constant depth H_∞ . The flow is characterized by a reference velocity $\mathcal{U}$ at the undisturbed free surface $z = 0$, and by a reference horizontal length λ . Although the free surface deformation in this case is null, the resulting full-depth stability problem cannot be solved by separation of variables, thus leading to a non-trivial structure in the corresponding eigensolutions.

We have shown that the surface-only model is capable of correctly reproducing both the eigenvalues and the eigenfunctions of the full-depth stability calculations up to moderate dimensionless flow speeds given by the Froude number $\mathcal{U}/\sqrt{g\lambda} = F \in \{0.25, 0.5, 1\}$. Moreover, we have shown the range of validity of our reduced model as a function of the water depth, and that good agreement is reached even down to an intermediate-deep water limit $H_\infty/\lambda \gtrsim 1$.

The second test problem concerns perturbations of a well-known analytical solution of the fully-nonlinear free surface Euler equations, namely the Gerstner waves solution. The flow associated to such waves is defined in Lagrangian coordinates and extends indefinitely below an upper non-flat free surface located at $z = H_0(x)$. The base flow used in our computations is defined by a one-parameter family of solutions dependent on kb_0 , where k is the wavenumber of the Gerstner wave and parameter $b_0 \leq 0$ adjusts the steepness of the base free surface elevation. We linearized the fully-nonlinear governing equations and boundary conditions around this background steady flow and study the resulting full-depth stability problem varying kb_0 . Even in this case the agreement between the exact and the model computations is remarkable both qualitatively and quantitatively for $kb_0 \lesssim -1$. For $kb_0 > -1$ the quantitative predictions of the surface-model start to deviate more significantly, although qualitatively the same trend as the full-depth calculations is still followed. Notably, the model accurately predicts the collision of the eigenvalues associated with the first mode and the resulting onset of the corresponding instability (figures 7(a) and 9), as well as the emergence of the Goldstone mode. In fact, as kb_0 approaches zero and the base flow becomes increasingly unstable, even the full-depth linearized Euler solution will suffer from the neglect of nonlinear terms, and hence the accuracy of the surface-only approximation becomes mute.

Our study provides the first evidence, to the best of our knowledge, that sufficiently steep Gerstner waves are unstable to two-dimensional perturbations. The stability of Gerstner waves was previously studied in three-dimensions under a short-wavelength assumption using a Lagrangian formalism (Leblanc 2004). Future research could extend our two-dimensional investigation using the closure condition presented here to study three-dimensional perturbations to Gerstner waves without making the short-wavelength assumption. In addition, our use of periodic boundary conditions here has restricted perturbations to have the same wavelength as the base Gerstner wave and therefore does not allow perturbations with longer or incommensurate wavelengths. Such perturbations could be investigated using a Floquet analysis, which we leave for future work.

In the authors' opinion, the most important direction for future work is to establish more clearly the conditions under which the closure condition, and hence the resulting surface-only model, loses either validity or usefulness. As discussed in §2.4, the closure should be viewed as a statement about the vertical structure of the surface-wave perturbations rather than as a general consequence of the governing equations. In the two test scenarios considered here, the closure performs remarkably well in deep water, but this could only be established by comparison with full-depth calculations. A more useful understanding of the model would therefore require identifying, in advance, the flow, parameter regimes, and types of perturbations for which its predictions can be expected to be accurate.

The limitations of the model are also clear from the information available to it. The only information about the vertical structure of the base flow entering the surface-only equations is its behaviour at the free surface, in particular the surface shear. Consequently, two background flows having the same surface values and surface shear but substantially different vertical structure in the interior will be indistinguishable to the surface-only model. For example, the shear-layer configuration considered by Longuet-Higgins (1998), in which the shear is confined to a layer below the free surface, cannot be distinguished

by our surface-only model from a base flow without such interior shear. Thus, the model cannot, in general, capture effects on surface waves arising from vorticity or shear confined away from the free surface. Understanding which features of the interior background flow can be neglected in determining the surface-wave dynamics would help establish when the closure can be expected to be valid, without requiring full-depth calculations for every case.

With these caveats, the present surface-only model could also be used to predict the behaviour of deep-water surface waves on complicated flows with strongly deformed free surfaces that are out of reach of fully three-dimensional simulations, such as a vortex dipole, as briefly discussed by Vivanco & Melo (2004) from an experimental point of view. Other extensions to the present work could include the generalization of the reduced model derived in this work to account for other physical mechanisms and regimes, such as the inclusion of capillary and dissipative effects, and its extension to weakly nonlinear and fully-nonlinear water waves. Finally, the model is a time-dependent, two-dimensional system that is not restricted to describing just time-periodic eigenmode solutions, and could be used in time-domain simulations. We envision it being used to investigate phenomena that would be extremely challenging to simulate using the full time-dependent Euler equations, for example, the scattering of surface waves by strong vortices in deep water or the time-dependent propagation of highly dispersive water waves, or even the interaction of surface waves with metamaterial-like periodic arrays generated by periodic base flows.

Supplementary material. Animations of the $b_0 = -0.5$ results of figure 8 are available in the supplementary material as Movie 1.

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Appendix A. Derivation of the closure condition without flow

We present here a simple argument due to Young (2025) that our closure condition holds in the simplest possible case of linear perturbations about a motionless, infinitely deep base flow. Let the perturbation velocity $\mathbf{u}'$ be given in terms of velocity potential ϕ' . Then the perturbation dynamics are governed by $\partial_t \phi' = -p'$, with $\nabla^2 \phi' = 0$. The combined kinematic and dynamic boundary condition on the free surface $z = H_0$ is $\partial_t p' = g w'$. In terms of the potential this becomes

$$\partial_{tt} \phi' + g \partial_z \phi' = 0. \quad (\text{A } 1)$$

Letting $\chi = \partial_{tt} \phi' + g \partial_z \phi'$, we have that $\chi = 0$ on H_0 by the boundary condition and $\nabla^2 \chi = 0$ in the bulk because ∇^2 commutes with ∂_t and ∂_z . As the fluid is infinitely deep, $\phi' \rightarrow$ a time-independent constant as $z \rightarrow -\infty$, and hence we have also $\chi \rightarrow 0$ as $z \rightarrow -\infty$. Hence, by uniqueness of the Dirichlet problem, $\chi = 0$ in the bulk. Thus, rather trivially, $\partial_z \chi = 0$ on $z = H_0$, which is the same as $\partial_z w' = \partial_t (\partial_z p'/g)$ on $z = H_0$, i.e. our closure condition (2.12) evaluated in this simplest case. Of course, by changing reference frame it can be shown that the closure condition also holds exactly for a base flow that is everywhere uniform. A more significant extension of this argument may work for an arbitrary potential base flow with a flat free surface, which we leave to future work.

512 Appendix B. Detailed derivation and analysis of the numerical solution to scenario 1

In section 3.1.1, we derived the equations for the modal pressure $p(y, z)$,

$$\begin{aligned} & \left(k\text{Fr} f_0(z)q_0(y) - \omega \right) \left(\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - k^2 p \right) \\ & - 2k\text{Fr} f_0(z)q'_0(y) \frac{\partial p}{\partial y} - 2k\text{Fr} f'_0(z)q_0(y) \frac{\partial p}{\partial z} = 0, \end{aligned} \quad (3.4a)$$

$$\frac{\partial p}{\partial z} - \omega^2 p + 2\omega k\text{Fr}q_0(y)p - k^2\text{Fr}^2 q_0^2(y)p = 0, \quad \text{on } z = 0, \quad (3.4b)$$

$$\frac{\partial p}{\partial z} = 0, \quad \text{on } z = -h_\infty, \quad (3.4c)$$

subject to periodic boundary conditions in y , with $f_0(z)$ and $q_0(y)$ known functions, Fr , h_∞ and k known constants, and ω an unknown eigenvalue. Here we solve equations (3.4) using a spectral collocation method. We expand the modal pressure $p(y, z)$ in Fourier and Chebyshev components along the spanwise and the vertical directions, respectively (Trefethen 2000). By using Fourier nodes along the y -coordinate we ensure periodic boundary conditions are automatically satisfied at the numerical level. To account for the vertical dependence and boundary conditions along the free surface using Chebyshev nodes, it is first necessary to remap the vertical domain from the physical one to $z_c \in [-1, 1]$. This is done by employing the following maps, dependent on whether the perturbation problem is solved in a finite-deep water domain ($h_\infty < \infty$), or in an infinitely deep-water one ($h_\infty = \infty$):

$$z(z_c) = \frac{h_\infty}{2}(z_c - 1), \quad h_\infty < \infty, \quad z(z_c) = B_{\text{map}} \tan \left[\frac{\pi}{4}(z_c - 1) \right], \quad h_\infty = \infty, \quad (\text{B } 1a,b)$$

The vertical gradient transforms according to

$$\frac{\partial}{\partial z} = \frac{2}{h_\infty} \frac{\partial}{\partial z_c}, \quad h_\infty < \infty, \quad \frac{\partial}{\partial z} = \left(\frac{4}{\pi B_{\text{map}}} \right) \frac{1}{1 + \tan^2 \left[\frac{\pi}{4}(z_c - 1) \right]} \frac{\partial}{\partial z_c}, \quad h_\infty = \infty. \quad (\text{B } 2a,b)$$

Calling D_y, D_z, D_y^2, D_z^2 the resulting numerical differentiation matrices in the two spatial directions, and the values of the eigenfunction p at the spectral nodes $\Phi_{i,j} = p(y_i, z_j)$ for $(i, j) \in [1, N_y] \times [1, N_z]$, equations (3.4) at the numerical level become

$$\left[(Fk f_0(z_j)q_0(y_i) - \omega)(D_y^2 + D_z^2 - k^2) - 2Fk f_0(z_j)q'_0(y_i)D_y - 2Fk f'_0(z_j)q_0(y_i)D_z \right] \Phi_{i,j} = 0, \quad (\text{B } 3a)$$

$$[D_z - \omega^2 + 2\omega Fk q_0(y_i) - F^2 k^2 q_0^2(y_i)] \Phi_{i,1} = 0, \quad (\text{B } 3b)$$

$$[D_z] \Phi_{i,N_z} = 0. \quad (\text{B } 3c)$$

The boundary conditions (B 3c) and (B 3b) are enforced directly in the bulk equation (B 3a) by finding the indices of the bulk matrices where $z_c = -1$ and $z_c = 1$, respectively, and substituting the corresponding discretized operators contained in the numerical boundary conditions. This yields a polynomial eigenvalue problem which can be compactly written in matrix form as

$$[K_N + \omega C_N + \omega^2 M_N] \Phi_N = \mathbf{0}, \quad (\text{B } 4)$$

513 where K_N, C_N, M_N represent the stiffness, damping and mass matrices coming from the implementation of
514 equations (B 3), and $\Phi_N = (\Phi_{1,1}, \dots, \Phi_{1,N_z}, \Phi_{2,1}, \dots, \Phi_{2,N_z}, \dots)$ is the total array of the eigenfunction $p(y, z)$
515 evaluated at the spatial nodes. This discretized problem is then solved numerically using MATLAB's `polyeig`
516 function.

517 It is well known that spectral methods are prone to giving rise to spurious eigenvalues and eigenfunctions.
518 These are numerical artifacts caused by numerical under-resolution of highly-oscillatory modes; that is,
519 eigenfunctions that do satisfy the discretized problem but do not approximate solutions of the continuous
520 eigenvalue problem. Following Brambley & Peake (2008) (see also Brambley 2007, p. 58–61), we implement
521 two tests to remove such spurious numerical modes from the numerical solution of equation (B 4). This is a
522 modified version of the numerical tests described in appendix A of Zuccoli *et al.* (2024) to account for the
523 presence of doubly-degenerate eigenvalues in the present case, as described in Zuccoli (2023, §2.4.1).

524 Appendix C. Unphysical modes of the model

We recall the modal formulation of the surface-only model equations (2.13) for the two scenarios investigated in this work. For scenario 1 these read

$$U_0|_{H_0} \partial_x \hat{\mathbf{u}}_\perp + \left(\hat{\mathbf{v}} \cdot (\partial_y U_0)|_{H_0} + \hat{\mathbf{w}} \cdot (\partial_z U_0)|_{H_0} \right) \mathbf{e}_x + \nabla_\perp \hat{h} = i\omega \hat{\mathbf{u}}_\perp, \quad U_0|_{H_0} \partial_x \hat{\mathbf{w}} + \hat{p}_z = i\omega \hat{\mathbf{w}}, \quad (\text{C } 1a,b)$$

$$U_0|_{H_0} \partial_x \hat{p}_z + \nabla_\perp \cdot \hat{\mathbf{u}}_\perp = i\omega \hat{p}_z, \quad U_0|_{H_0} \partial_x \hat{h} - \hat{\mathbf{w}} = i\omega \hat{h}, \quad (\text{C } 1c,d)$$

525 with ω denoting eigenvalues and $\mathbf{w} = (\hat{\mathbf{u}}_\perp, \hat{\mathbf{w}}, \hat{p}_z, \hat{h})^T$ the corresponding eigenfunctions.

For scenario 2, the surface-only model equations read

$$U_0|_{H_0} \partial_x \hat{u} + \hat{u} \partial_x U_0|_{H_0} + \hat{\mathbf{w}} \cdot \partial_z U_0|_{H_0} + \partial_x (g_0 \hat{h}) - H'_0 \hat{p}_z = i\omega \hat{u}, \quad (\text{C } 2a)$$

$$U_0|_{H_0} \partial_x \hat{\mathbf{w}} + \hat{\mathbf{u}} \cdot \partial_x W_0|_{H_0} + \hat{\mathbf{w}} \cdot \partial_z W_0|_{H_0} + \hat{p}_z = i\omega \hat{\mathbf{w}}, \quad (\text{C } 2b)$$

$$U_0|_{H_0} \partial_x (\hat{p}_z / g_0) - \frac{\varphi_0}{g_0} \hat{p}_z + \partial_x \hat{u} = i\omega \hat{p}_z / g_0, \quad (\text{C } 2c)$$

$$U_0|_{H_0} \partial_x \hat{h} + \hat{u} H'_0 - \varphi_0 \hat{h} - \hat{\mathbf{w}} = i\omega \hat{h}, \quad (\text{C } 2d)$$

526 with again ω denoting eigenvalues and $\mathbf{w} = (\hat{u}, \hat{\mathbf{w}}, \hat{p}_z, \hat{h})^T$ the corresponding eigenfunctions.

527 The spectrum of the surface-model equations contains “unphysical” eigenvalues that need to be removed.
528 Such unphysical modes arise because the surface-model equations cannot distinguish whether the fluid extends
529 infinitely deep below or infinitely high above the base free surface H_0 .

To decide which eigenvalues are unphysical and need to be discarded, we define the following parameter

$$\beta = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left(\frac{\partial_z p'|_{H_0(s)}}{p'|_{H_0(s)}} \right) ds = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left(\frac{\hat{p}_z(s)}{g_0(s) \hat{h}(s)} \right) ds, \quad (\text{C } 3)$$

where the integration variable s here refers to y in the first scenario, and x in the second scenario. Parameter β can be interpreted as a mean value of the vertical decay/growth rate of the modes. Thus, being interested in the propagation of surface waves on the upper surface above an infinitely deep fluid, if β is sufficiently positive, the mode is likely to be a good candidate. On the other hand, if β is sufficiently negative, the mode is likely to be “unphysical” and needs to be discarded. To evaluate this feature more quantitatively, we set a tolerance δ_β and define the following criteria to recognize “physical” meaningful eigenvalues from “unphysical” ones:

$$\begin{aligned} \text{if } \beta > \delta_\beta &\Rightarrow \text{physical eigenvalue,} \\ \text{if } \beta < -\delta_\beta &\Rightarrow \text{unphysical eigenvalue} \end{aligned} \quad (\text{C } 4)$$

For scenario 1, the expression of β is obtained analytically as

$$\beta = \omega^2 + \frac{k^2 \text{Fr}^2}{2}. \quad (\text{C } 5)$$

In figure 11 we show the whole spectrum of the surface-only model equations (C 1)–(C 2) computed at $\text{Fr} = 0.5$ for the scenario 1, and at $b_0 = -1$ for the scenario 2, both with tolerances $\delta_\beta = 0.01$. The plots show the eigenvalues passing the selection criteria (green dots) and those which do not pass it (red crosses). More specifically, regarding the parabolic branches in panel (b) of figure 11 these can be thought to be the continuation as b_0 gets larger of the unphysical branches obtained when $b_0 \rightarrow -\infty$. In this limit, the eigenvalues predicted by the model are given by

$$\omega_n^{(1,2)} = n \pm \sqrt{|n|}, \quad \text{and} \quad \omega_n^{(3,4)} = n \pm i\sqrt{|n|}, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (\text{C } 6)$$

530 The two branches $\omega^{(3,4)}$ correspond to the “unphysical” eigenvalues with $\beta = -1$.

531 Appendix D. Exact solutions of the eigenvalue problems in the case of a flat free surface

532 In this section we want to obtain the exact eigensolutions for both the first model validation problem and the
533 Gerstner wave perturbation problem in the limit the steady free surface becomes flat. In the first case this
534 condition corresponds to $\text{Fr} = 0$; in the second it corresponds to $b_0 \rightarrow \infty$. For the purposes of this paper, we
535 show here the solutions in the infinitely deep-water regime.

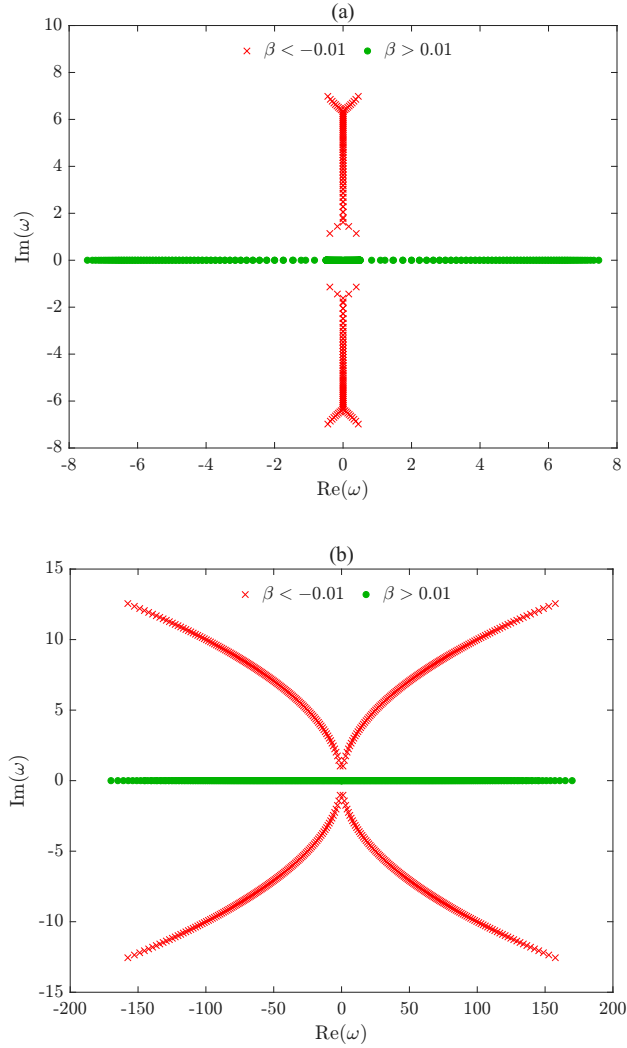


Figure 11. (a): Full spectrum of the model equations (C 1) computed at $Fr = 0.5$. (b): Full spectrum of the model equations (C 2) computed at $b_0 = -1$. Red crosses: “unphysical” eigenvalues not passing the selection criteria (C 4). Green dots: “physical” eigenvalues passing the selection criteria (C 4).

536

D.1. Exact solution for scenario 1

When $Fr = 0$ and $h_\infty = \infty$, the eigenproblem (3.4) simplifies into

$$\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - k^2 p = 0, \quad (\text{D } 1a)$$

$$\frac{\partial p}{\partial z} = \omega^2 p, \quad \text{on } z = 0, \quad \text{and} \quad p \rightarrow 0, \quad \text{as } z \rightarrow -\infty. \quad (\text{D } 1b,c)$$

Due to the periodicity in y , we can search for a solution in the form $p(y, z) = f(z)e^{iny}$, with $n \in \mathbb{Z}$. This gives the eigenproblem

$$f'' - (n^2 + k^2)f = 0, \quad (\text{D } 2a)$$

$$f'(0) = \omega^2 f(0), \quad \text{and} \quad f(-\infty) = 0, \quad (\text{D } 2b,c)$$

whose solutions read

$$f(z) = Ae^{\sqrt{n^2+k^2}z}, \quad \text{and} \quad \omega_n = \pm(n^2 + k^2)^{1/4}, \quad n \in \mathbb{Z}. \quad (\text{D } 3a,b)$$

D.2. Exact solution for scenario 2

When $b_0 = -\infty$, the velocity is uniform in the x -direction only and all the quantities depending on b_0 go to zero. Hence, the eigenproblem (4.6) can be re-written entirely in terms of the pressure and assumes the following form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0, \quad (\text{D } 4a)$$

$$\frac{\partial p}{\partial z} = -\left(\frac{\partial}{\partial x} - i\omega\right)^2 p, \quad \text{on } z = 0, \quad \text{and} \quad p \rightarrow 0, \quad \text{as } z \rightarrow -\infty. \quad (\text{D } 4b,c)$$

Again, due to the periodicity in x , we can search for a solution in the form $p(y, z) = \psi(z)e^{inx}$, with n being an integer. This returns the eigenproblem

$$\psi'' - n^2\psi = 0, \quad (\text{D } 5a)$$

$$\psi'(0) = (\omega - n)^2\psi(0), \quad \text{and} \quad \psi(-\infty) = 0, \quad (\text{D } 5b,c)$$

whose solutions read

$$\psi(z) = e^{|n|z}, \quad \text{and} \quad \omega_n = n \pm \sqrt{|n|}, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (\text{D } 6a,b)$$

Note that the convention used to denote the eigenvalues computed numerically and shown in figure 7 is that of increasing modulus, i.e. $|\omega_0| < |\omega_{\pm 1}| < |\omega_{\pm 2}| \dots$ and does not correspond to the notation used in equation (D 6), which instead denotes the eigenvalues based on the periodicity of the corresponding eigenfunctions.

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